P1 Chapter 5 :: Straight Line Graphs

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Chapter Overview

There is little new theory since GCSE, but the algebraic manipulation is harder.

1:: $y = mx + c$, Gradient & Determining Equations

Find the equation of the line passing through $(2,3)$ and $(7,5)$, giving your equation in the form $ax + by + c = 0$, where a, b, c are integers.

NEW! since GCSE
The equation $y - y_1 = m(x - x_1)$ for a line with given gradient and going through a given point.

2:: Parallel/Perpendicular Lines

A line is perpendicular to $3x + 8y - 11 = 0$ and passes through $(0, -8)$. Find the equation of the line.

3:: Lengths and Areas

The line $2x + 3y = 6$ crosses the x-axis and y-axis at the points A and B respectively. Determine:
(a) The length AB and
(b) The area OAB.

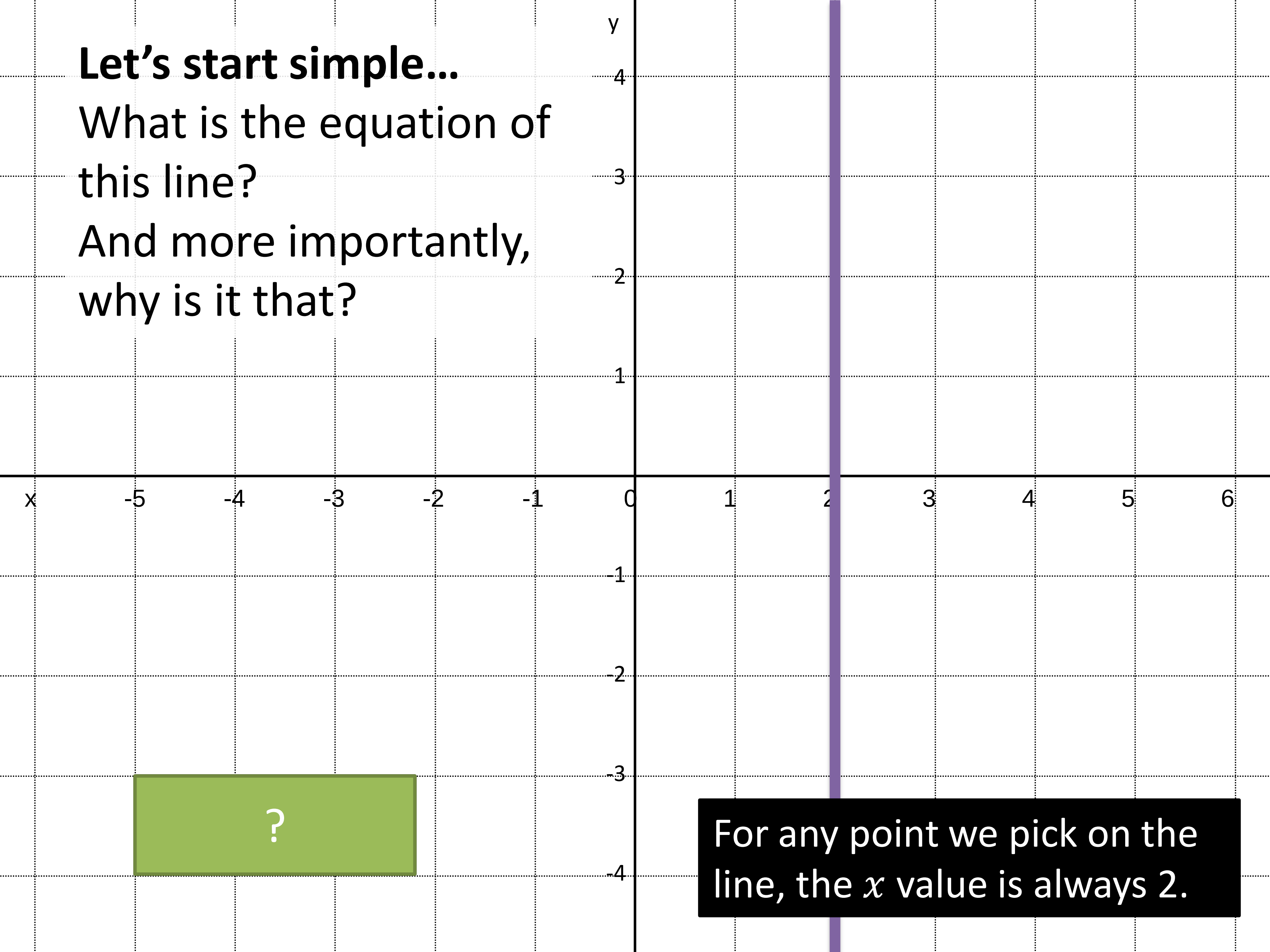
4:: Modelling

A plumber charges a fixed cost plus a unit cost per day. If he charges £840 for 2 days work and …

Let’s start simple…

What is the equation of this line?

And more importantly, why is it that?

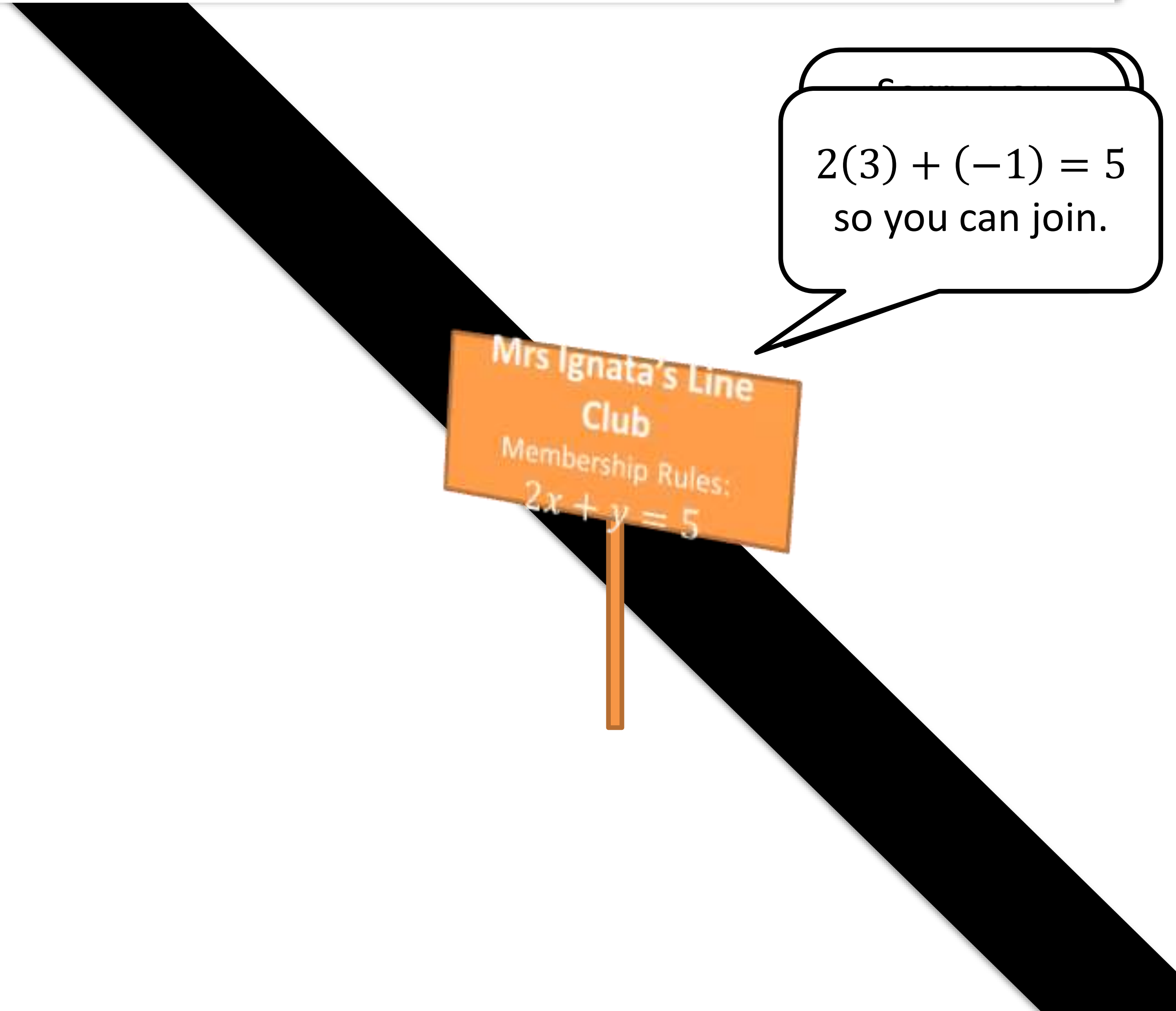


?

For any point we pick on the line, the x value is always 2.

Lines and Equations of Lines

✎ A line consists of all points which satisfy some equation in terms of x and/or y.



Examples

This means we can **substitute** the values of a coordinate into our equation whenever we know the point lies on the line.

The point $(5, a)$ lies on the line with equation $y = 3x + 2$. Determine the value of a.

?

Find the coordinate of the point where the line $2x + y = 5$ cuts the x-axis.

?

Test Your Understanding

Determine where the line $x + 2y = 3$ crosses the:

a) y-axis:

?

b) x-axis:

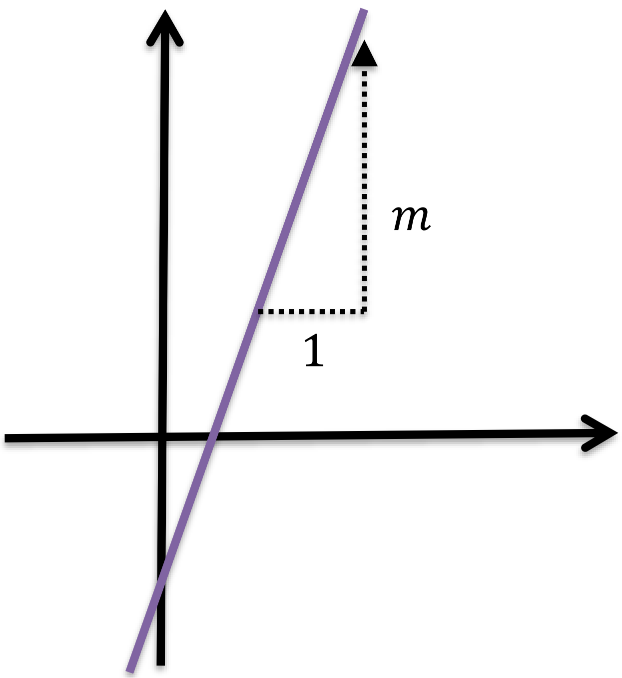
?

What mistakes do you think it's easy to make?

- ?
-

Recap of gradient

The steepness of a line is known as the **gradient**.
It tells us what y changes by as x increases by 1.



So if the y value increased by 6 as the x value increased by 2, what is y increasing by for each unit increase of x?
How would that give us a suitable formula for the gradient m?

$$m = ?$$

Δ is the (capital) Greek letter "delta" and means **"change in"**.

Textbook Note:

The textbook uses $m = \frac{y_2 - y_1}{x_2 - x_1}$ for two points (x_1, y_1) and (x_2, y_2). Reasons I don't use it for non-algebraic coordinates:

- Students often get the y_1 and y_2 the wrong way round (or with the x's)
- Students often make sign errors when dealing with negatives, e.g. $(-3) - (-4)$
- It can't be done as easily mentally,
- Students see it as "yet another formula to learn" when really all you need is to appreciate is what gradient is, i.e. "y change per x change".

Examples

Find the gradient of the line that goes through the points:

1 $(1, 4) \quad (3, 10)$ $m =$?

2 $(5, 7) \quad (8, 1)$ $m =$?

3 $(2, 2) \quad (-1, 10)$ $m =$?

4 Show that the points $A(3,4), B(5,5), C(11,8)$ all lie on a straight line.

?

Further Example

The line joining $(2, -5)$ to $(4, a)$ has gradient -1. Work out the value of a.

?

$y = mx + c$

Every linear equation can be written in the form:

$$y = \underbrace{m}_{\text{Gradient}}x + \underbrace{c}_{y\text{-intercept}}$$

Why does it work?

?

$y = mx + c$

Determine the gradient and y-intercept of the line with equation $4x - 3y + 5 = 0$

?

Make y the subject so we have the form
$$y = mx + c$$
Put y on the side it's positive.

Divide **each** term by 3; don't write $y = \frac{4x+5}{3}$ otherwise it's not in the form $y = mx + c$

This is algebra, so use improper fractions, and not mixed numbers or recurring decimals.

$ax + by + c = 0$

At GCSE, $\boldsymbol{y = mx + c}$ was the main form you would express a straight line equation, sometimes known as the '**slope-intercept form**'.
But another common form is $\boldsymbol{ax + by + c = 0}$, where a, b, c are integers.
This is known as the '**standard**' form.

Express $y = \frac{1}{3}x - \frac{2}{3}$ in the form $ax + by + c = 0$, where a, b, c are integers.

?

We'll see on the next slide WHY we might want to put an equation in this form over $y = mx + c$...

Just for your interest...

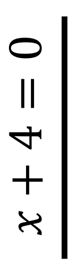
Why might we want to put a straight line equation in the form $ax + by + c = 0$?

$y = mx + c$
"Slope-Intercept Form"

$ax + by + c = 0$
"Standard Form"

Coverage

$y = mx + c$ doesn't allow you to represent vertical lines. Standard form allows us to do this by just making b zero.



Symmetry

In general, the '**linear combination**' of two variables x and y is $ax + by$, i.e. "some amount of x and some amount of y". There is a greater elegance and symmetry to this form over $y = mx + c$ because x and y appear similarly within the expression.

Usefulness

This more 'elegant' form also means it ties in with vectors and matrices. In FM, you will learn about the '**dot product**' of two vectors:

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = ax + by$$

thus since $ax + by + c = 0$, we can represent a straight line using:

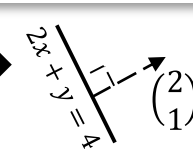
$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + c = 0 \qquad (1)$$

We can extend to 3D points to get the equation of a **plane**:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + d = 0 \qquad (2)$$

Conveniently, in equation (1), the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is **perpendicular to the line**. And in equation (2), the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is perpendicular to the plane. Nice!

$$2x + y = 4 \rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 4 \rightarrow$$



Test Your Understanding

Express $y = \frac{2}{5}x + \frac{3}{5}$ in the form $ax + by + c = 0$, where a, b, c are integers.

?

Exercise 5A/5B

Pearson Pure Mathematics Year 1/AS

Pages 90-91, 92-93

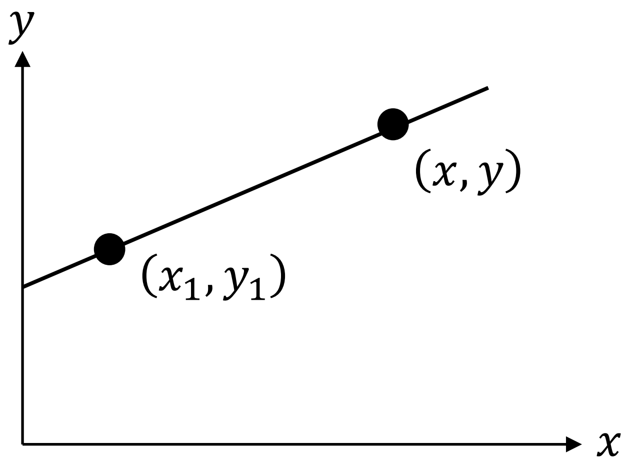
Equations using one point + gradient

Find the equation of the line that goes through $(3,5)$ and has gradient 2.

How would you have done this at **GCSE**?

?

A new way...



Note: Note that x_1 and y_1 are <u>constants</u> while x and y are <u>variables</u>. The latter are variables because as these 'vary', we get different points on the line.

Suppose that (x_1, y_1) is some fixed point on the line that we specify (e.g. $(3,5)$). Suppose that (x, y) represents a generic point on the line, which is allowed to change as we consider different points on this line.

Then:

$m =$?

Thus:

The equation of a line that has gradient m and passes through a point (x_1, y_1) is:

$$y - y_1 = m(x - x_1)$$

Let's revisit:

Find the equation of the line that goes through $(3,5)$ and has gradient 2.

?

Quickfire Questions

In a nutshell: You can use this formula whenever you have (a) a gradient and (b) any point on the line.

Gradient	Point	(Unsimplified) Equation
3	$(1,2)$	?
5	$(3,0)$	?
2	$(-3,4)$	?
$\frac{1}{2}$	$(1,-5)$	?
9	$(-4,-4)$	?

Important Side Note: I've found that many students shun this formula and just use the GCSE method. Please persist with it – it'll be much easier when fractions are involved. Further Mathematicians, <u>don't even think</u> about using the GCSE method, because you'll encounter massive headaches when you consider algebraic points. Trust me on this one!

Using 2 points

Find the equation of the line that goes through $(4,5)$ and $(6,2)$, giving your equation in the form $ax + by + c = 0$.

?

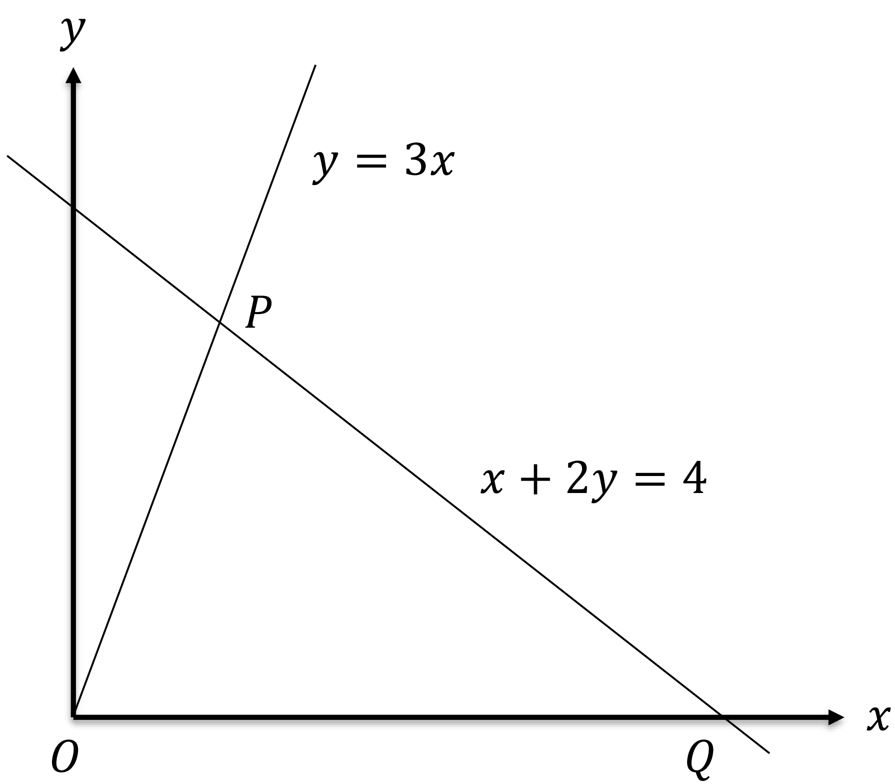
Test Your Understanding:
Find the equation of the line that goes through $(-1,9)$ and $(4,5)$, giving your equation in the form $ax + by + c = 0$.

?

Exercise 5C

Pearson Pure Mathematics Year 1/AS
Pages 94-95

Intersection of lines



We can find the point of intersection of 2 lines by solving simultaneously.

The diagram shows two lines with equations $y = 3x$ and $x + 2y = 4$, which intersect at the point P.

a) Determine the coordinates of P.

?

b) The line $x + 2y = 4$ intersects the x-axis at the point Q. Determine the coordinate of Q.

?

Test Your Understanding

C1 Edexcel May 2013 Q6

The straight line L_1 passes through the points $(-1, 3)$ and $(11, 12)$.

(a) Find an equation for L_1 in the form $ax + by + c = 0$, where a, b and c are integers.

(4)

The line L_2 has equation $3y + 4x - 30 = 0$.

(b) Find the coordinates of the point of intersection of L_1 and L_2.

(3)

(a) ?

(b) ?

Exercise 5D

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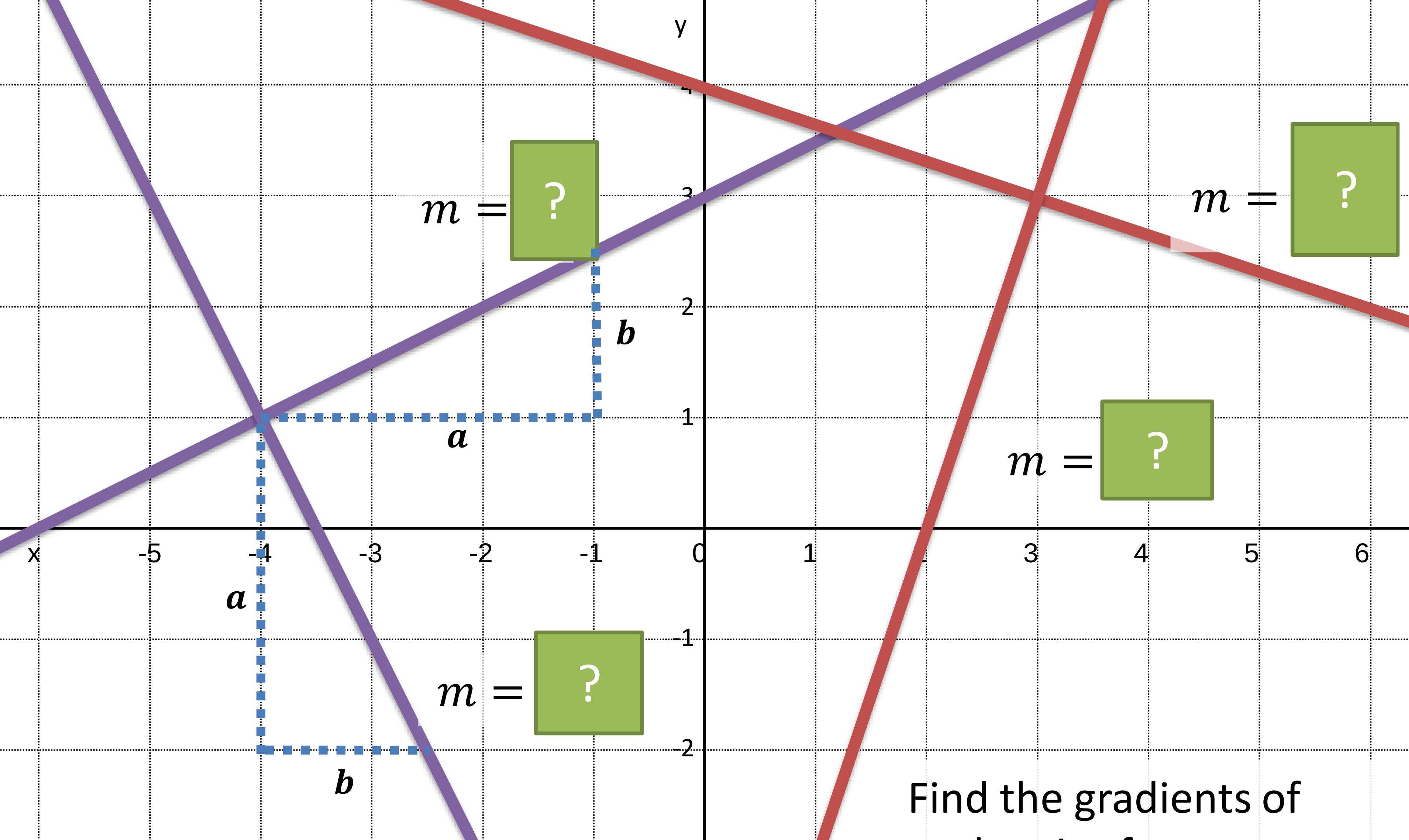


Using the changes above, we can see the gradient of one line is $m = \frac{b}{a}$ and the other $m = \frac{b}{-a}$. One is the 'negative reciprocal' of the other.

Find the gradients of each pair of **perpendicular** lines. What do you notice?

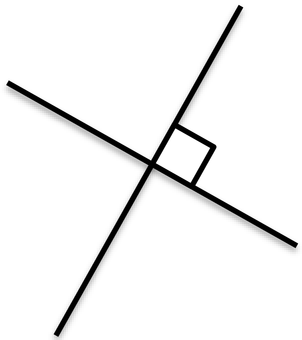
Perpendicular Lines

- The gradients of parallel lines are equal.
- If two lines are perpendicular, then the gradient of one is the **negative reciprocal** of the other.

$$m_1 = -\frac{1}{m_2}$$

- To <u>**show**</u> that two lines are perpendicular:

$$m_1 m_2 = -1$$



Gradient	Gradient of Perpendicular Line
2	?
-3	?
$\frac{1}{4}$	?
5	?
$-\frac{2}{7}$	?
$\frac{7}{5}$	?

Example Problems

1. A line is goes through the point (9,10) and is perpendicular to another line with equation $y = 3x + 2$. What is the equation of the line?

?

2. A line L_1 goes through the points $A(1,3)$ and $B(3,-1)$. A second line L_2 is perpendicular to L_1 and passes through point B. Where does L_2 cross the x-axis?

?

3. Are the following lines parallel, perpendicular, or neither?

$$y = \frac{1}{2}x$$
$$2x - y + 4 = 0$$

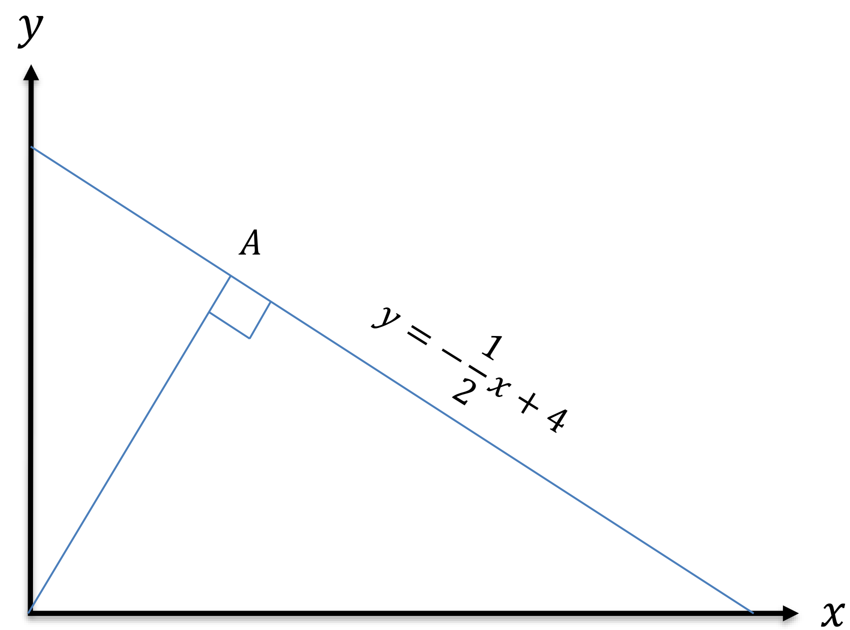
?

Test Your Understanding

1 A line goes through the point (4,7) and is perpendicular to another line with equation $y = 2x + 2$. What is the equation of the line? Put your answer in the form $ax + by + c = 0$, where a, b, c are integers.

?

2 Determine the point A.



?

Exercise 5E/5F

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Extension Problems

1 *[MAT 2004 1D]*
What is the reflection of the point $(3,4)$ in the line $3x + 4y = 50$?

?

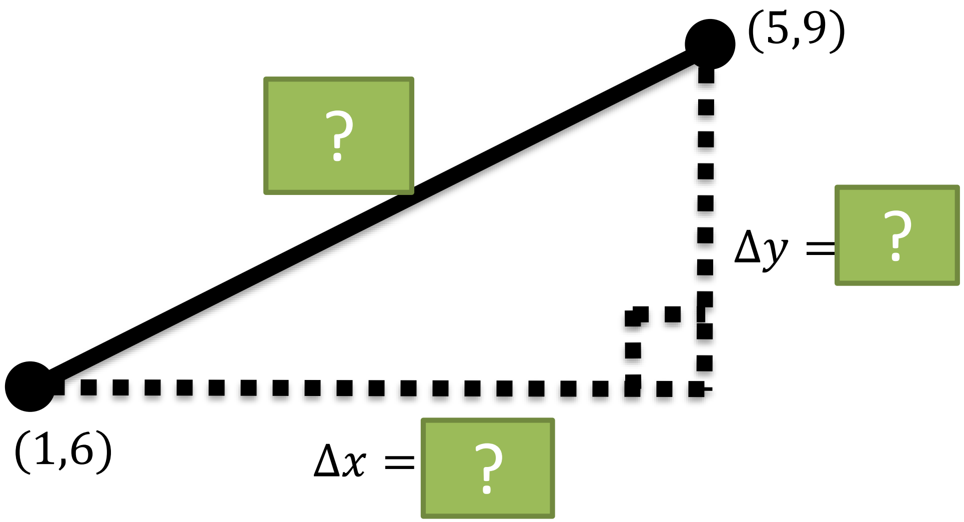
2 *[MAT 2014 1D]* The reflection of the point $(1,0)$ in the line $y = mx$ has coordinates: (in terms of m)

?

3 *[STEP I 2004 Q6]* The three points A, B, C have coordinates $(p_1, q_1), (p_2, q_2)$ and (p_3, q_3), respectively. Find the point of intersection of the line joining A to the midpoint of BC, and the line joining B to the midpoint of AC. Verify that this point lies on the line joining C to the midpoint of AB.
The point H has coordinates $(p_1 + p_2 + p_3, q_1 + q_2 +$

Distances between points

Recall: Δ (said 'delta') means "change in".



How could we find the **distance** between these two points?

Hint: ?

?

Distance between two points:

$$\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Examples

Distance between:

$(3,4)$ and $(5,7)$
$(5,1)$ and $(6,-3)$
$(0,-2)$ and $(-1,3)$

?
?
?

Note: Unlike with gradient, we don't care if the difference is positive or negative (it's being squared to make it positive anyway!)

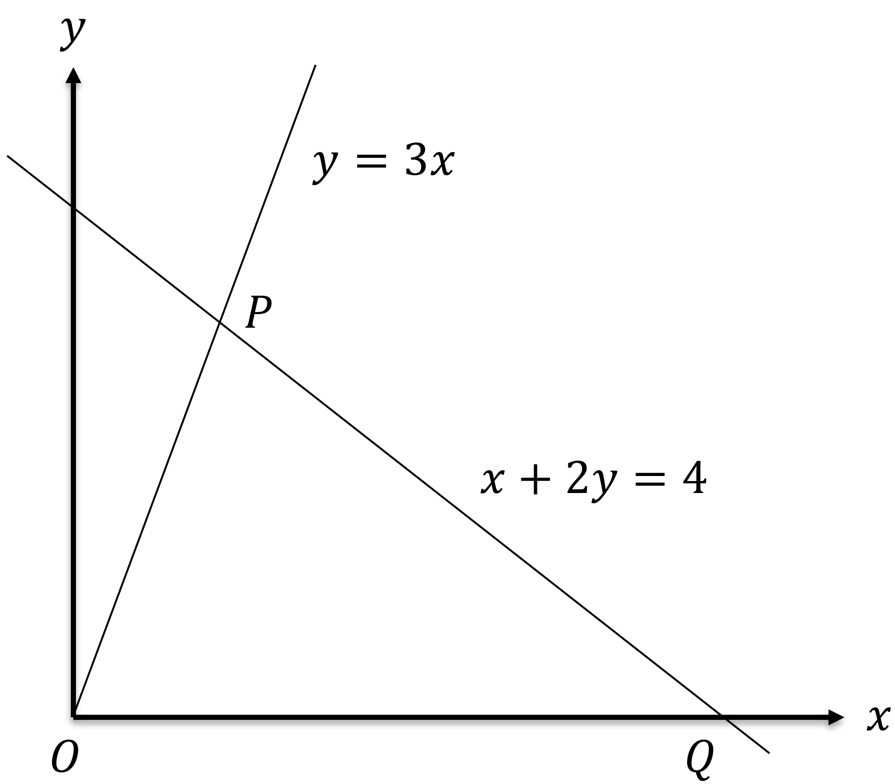
Quickfire Questions:

Distance between:

$(1,10)$ and $(4,14)$
$(3,-1)$ and $(0,1)$
$(-4,-2)$ and $(-12,4)$

?
?
?

Area of Shapes



The diagram shows two lines with equations $y = 3x$ and $x + 2y = 4$, which intersect at the point P.

a) Determine the coordinates of P.

(We did this in a previous lesson)

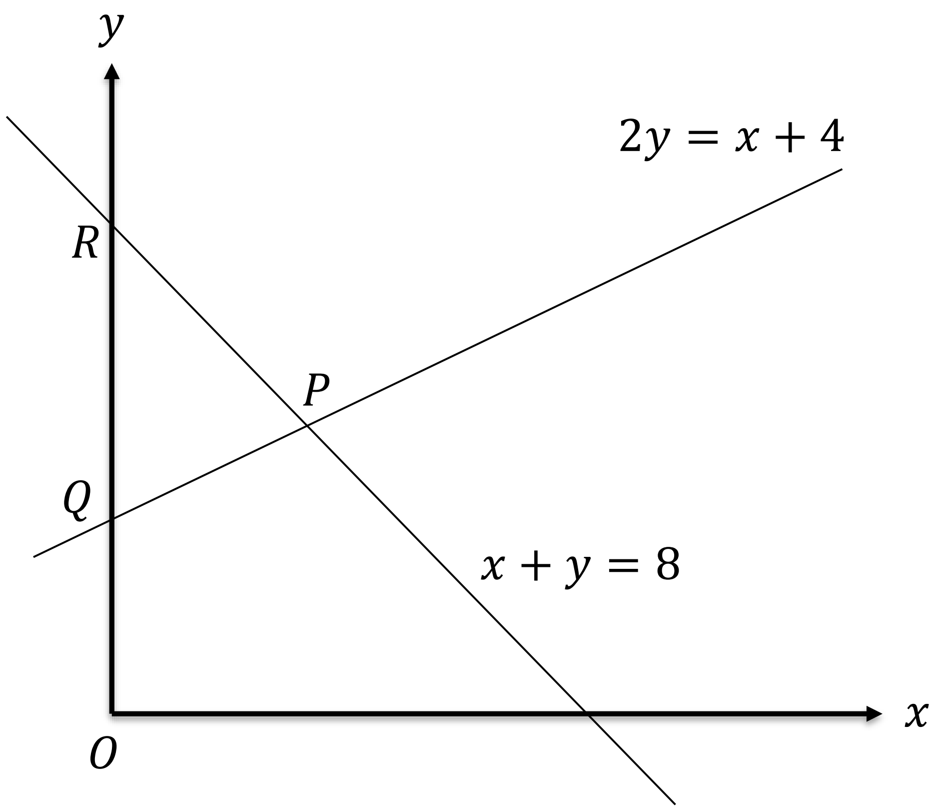
Just solve two equations simultaneously.

$$\boldsymbol{x + 2(3x) = 4}$$
$$\boldsymbol{7x = 4}$$
$$\boldsymbol{x = \frac{4}{7}}$$
$$\boldsymbol{y = 3\left(\frac{4}{7}\right) = \frac{12}{7}}$$

b) The line $x + 2y = 4$ intersects the x-axis at the point Q. Determine the area of the triangle OPQ.

?

Further Example



Tip: When finding areas of triangles in exam questions, one line is often vertical or horizontal. You should generally choose this to be the 'base' of your triangle.

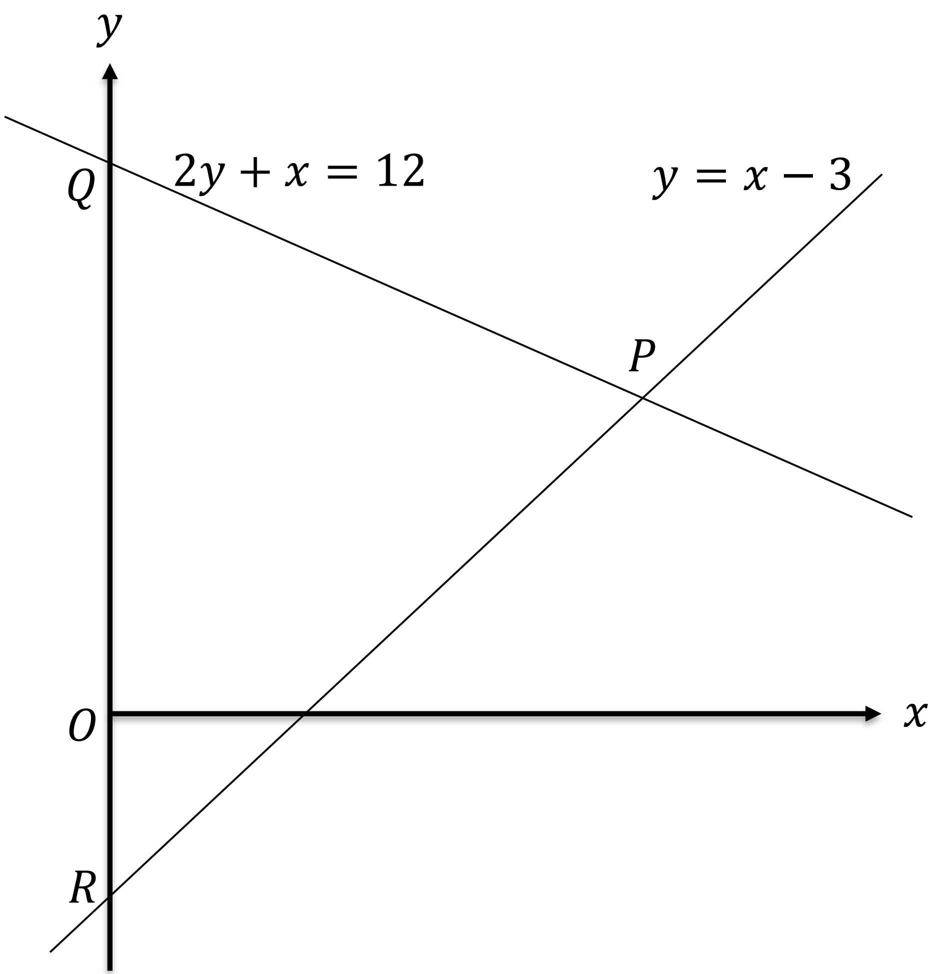
a Determine the length of PQ.

?

b Determine the area PQR.

?

Test Your Understanding



a) Determine the coordinate of P.

?

b) Determine the area of PQR.

?

c) Determine the length PQ.

?

Exercise 5G

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Page 102-103

Extension Problems

1 *[MAT 2001 1C]*
The shortest distance from the origin to the line $3x + 4y = 25$ is what?

?

What’s the point of straight line equations?

We saw in Chapter 2 that lots of things in real life have a ‘quadratic’ relationship, e.g. vertical height with time. Lots of real life variables have a ‘linear’ relationship, i.e. **there is a fixed increase/decrease in one variable each time the other variable goes up by 1 unit.**

Examples

Car sales made and take home pay.

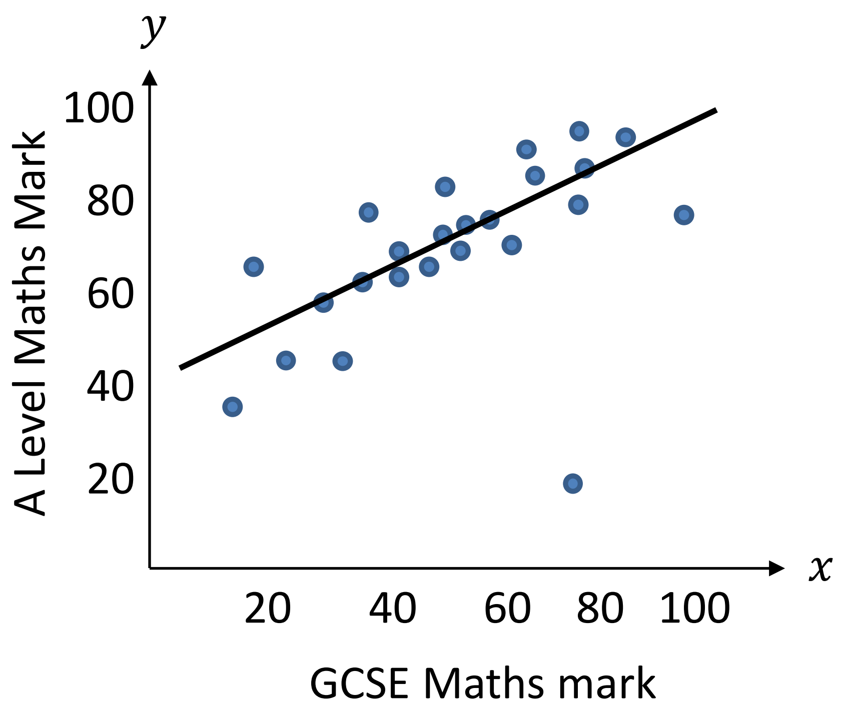
The relationship between Celsius and Fahrenheit.

Temperature and altitude (in a particular location)

(And a pure maths one:) The nth term of an arithmetic series.

2, 5, 8, 11, 14, …

Modelling



A mathematical model is an attempt to model a real-life situation based on mathematical concepts.

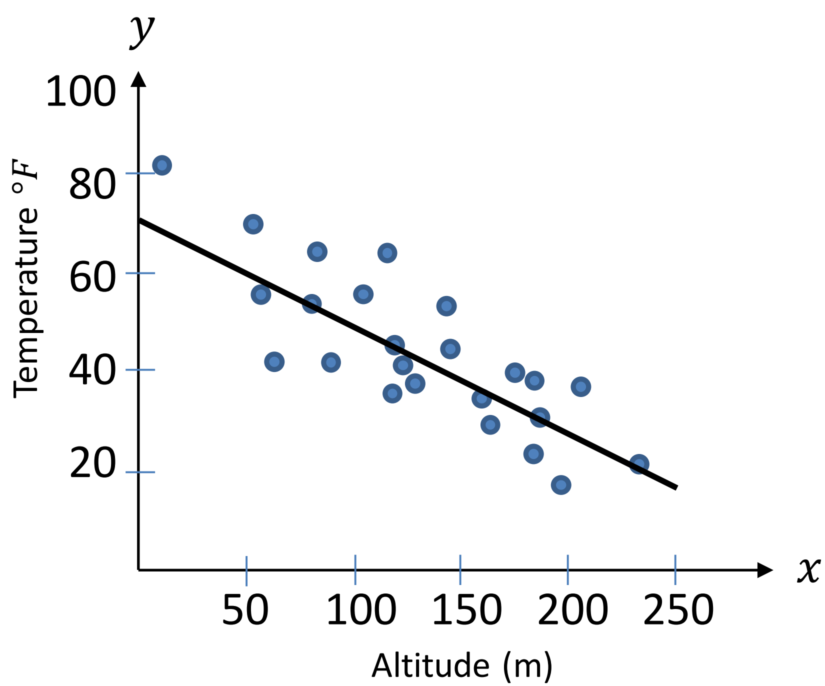
For this example, **our model might be a linear model with equation $\boldsymbol{y = mx + c}$** where x is a student's GCSE mark and y is the predicted A Level mark.

Such a linear model can be drawn as a line of best fit.
The data obviously doesn't fit this line exactly. This chosen model may only partially fit the data (and the further the points are away from the line, the less suitable this model is).
We might decide another model, e.g. $y = ax^2 + bx + c$, is more appropriate.

But if we choose a well-known model such as a linear one, then we can use **established mathematical theory** in useful ways:

- We need to choose the most appropriate 'parameters' m and c so the model best matches the data. You will learn in S1 there are existing techniques to do this.
- We can then predict a student's A Level mark based on their maths mark.

Example



The temperature y at different points on a mountain is recorded at different altitudes x.
Suppose we were to use a linear model $y = mx + c$.

---

a Determine m and c (you can assume the line goes through $(0,70)$ and $(250,20)$.

?

b **<u>Interpret</u>** the meaning of m and c **<u>in this context</u>**.

?

c Predict at what altitude the temperature reaches $0°F$.

?

Evaluating if a linear model is most suitable

Choosing a model for our data usually involves making **simplifying assumptions**. For a linear model we are assuming:

The y value goes up the same amount for each unit increase in x.

The current population of Bickerstonia is 26000. This year (2017) the population increased by 150. Matt decides to model the population P based on the years t after 2017 by the linear model:

$$P = mt + c$$

Why might this not be a suitable model?

?

Exercise 5H

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