P1 Chapter 12 :: Differentiation

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Chapter Overview

Those who have done either IGCSE Mathematics, IGCSE Further Mathematics or Additional Mathematics would have encountered this content. Otherwise it will be completely new!

1:: First Principles and finding the derivative of polynomials.

If $y = 3x^2 + \sqrt{x}$, find $\frac{dy}{dx}$

2:: Find equations of tangents and normal to curves.

The point $P(3,9)$ lies on the curve C with equation $y = x^2$. Determine the equation of the tangent to C at the point P.

3:: Identify increasing and decreasing functions.

Find the range of values for which $f(x) = x^3 - x$ is increasing.

4:: Find and understand the second derivative $\frac{d^2y}{dx^2}$ or $f''(x)$

If $y = x^4 - 3x^2$ determine $\frac{d^2y}{dx^2}$

5:: Find stationary points and determine their nature.

Find the stationary points of $y = x^3 - x$ and state whether each is a maximum or minimum point.

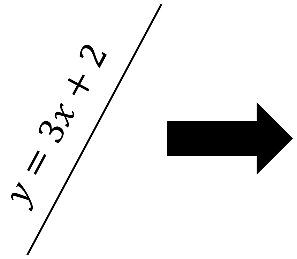
6:: Sketch a gradient function.

Draw $y = x^3$ and its gradient function on the same axes.

7:: Model real-life problems.

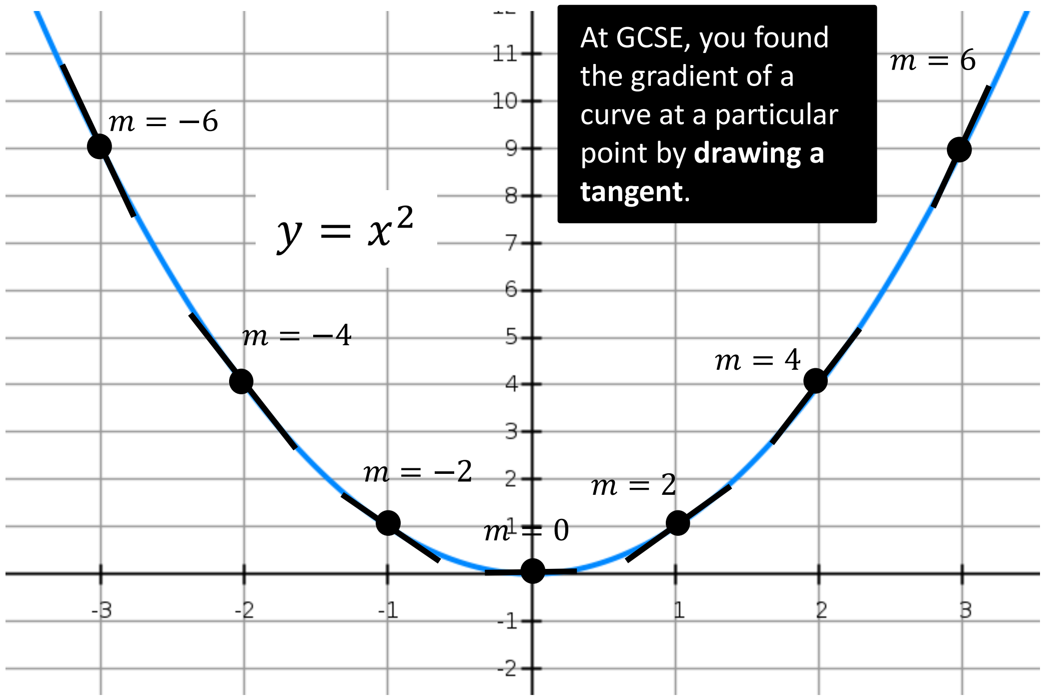
Gradient Function

For a straight line, the gradient is **constant**:



$m = 3$

However, for a curve **the gradient varies**. We can no longer have a single value for the gradient; **we ideally want an expression in terms of x** that gives us the gradient for any value of x (unsurprisingly known as the **<u>gradient function</u>**).



At GCSE, you found the gradient of a curve at a particular point by **drawing a tangent**.

x	-3	-2	-1	0	1	2	3
Gradient	?	?	?	?	?	?	?

By looking at the relationship between x and the gradient at that point, can you come up with an expression, in terms of x for the gradient?

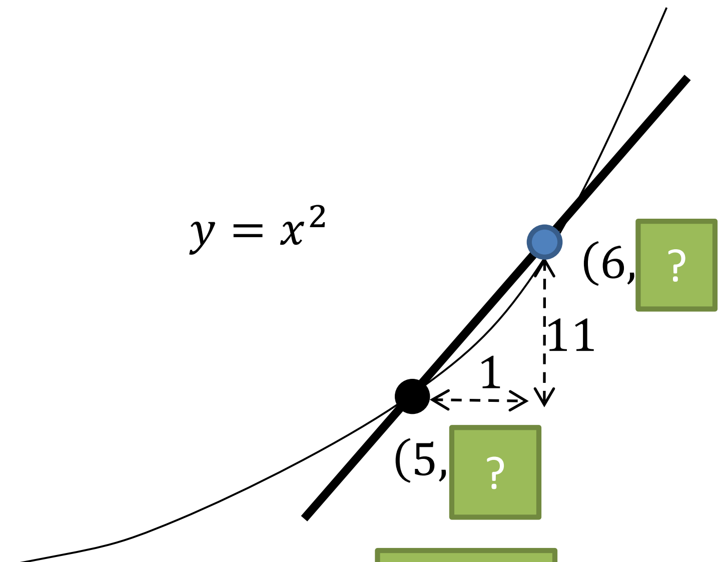
?

Finding the Gradient Function

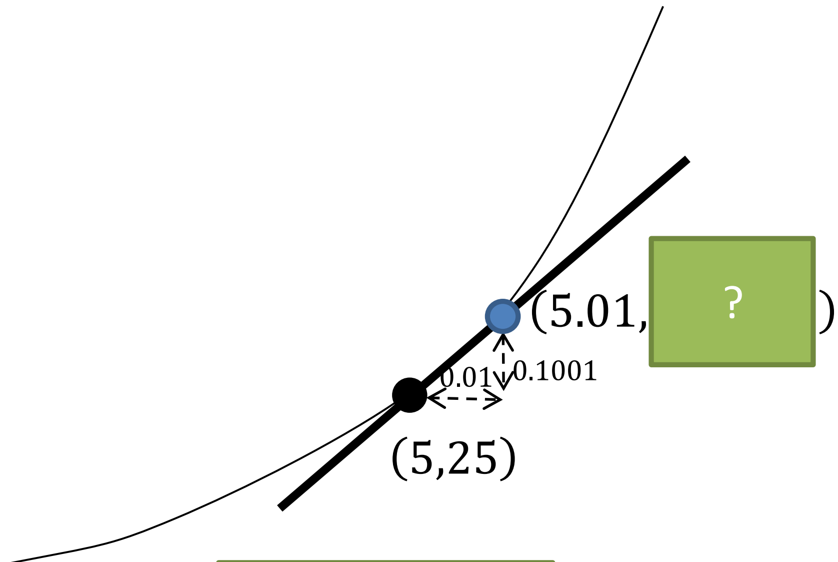
The question is then: Is there a method to work out the gradient function without having to draw lots of tangents and hoping that we can spot the rule?

To approximate the gradient on the curve $y = x^2$ when $x = 5$, we could pick a point on the curve just slightly to the right, then find the gradient between the two points:

As the second point gets closer and closer, the gradient becomes a better approximation of the true gradient:



$m = \frac{\Delta y}{\Delta x} =$?

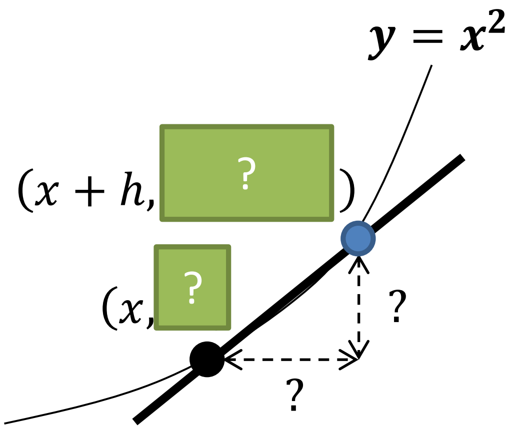


$m =$?

The actual gradient when $x = 5$ is 10, so this approximation is damn close!

Finding the Gradient Function

This gives us a numerical method to get the gradient **at a particular $\boldsymbol{x}$**, but doesn't give us the gradient function in general. Let's use exactly the same method, but keep x general, and make the 'small change' (which was previously 0.01) 'h':



$$gradient = \boxed{?}$$
$$= \boxed{?}$$
$$= \boxed{?}$$
$$= \boxed{?}$$

As always, gradient is change in y over change in x.

The lim means "*the limit of the following expression as h tends towards 0*".
For example, $\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$, because as x "tends towards" infinity, the "limiting" value of the expression is 0.

The h disappears as h tends towards 0, i.e. we can effectively treat it as 0 at this point.

And voila, we got the $2x$ we saw earlier!

Finding the Gradient Function

✏ The gradient function, or derivative, of the curve $y = f(x)$ is written as $f'(x)$ or $\frac{dy}{dx}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The gradient function can be used to find the gradient of the curve for any value of x.

We will soon see an easier/quicker way to differentiate expressions like $y = x^3 - x$ without using 'limits'. But this method, known as **differentiating by first principles**, is now in the A Level syllabus, and you could be tested on it!

Notation Note:

Whether we use $\frac{dy}{dx}$ or $f'(x)$ for the gradient function depends on whether we use $y =$ or $f(x) =$ to start with:

$$y = x^2 \quad \rightarrow \quad \frac{dy}{dx} = 2x$$

"Leibniz's notation"

$$f(x) = x^2 \quad \rightarrow \quad f'(x) = 2x$$

"Lagrange's notation"

Finding the Gradient Function

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$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The gradient function can be used to find the gradient of the curve for any value of x.

There's in fact a third way to indicate the gradient function, notation used by Newton: (but not used at A Level)

$$y = x^2 \quad \rightarrow \quad \dot{y} = 2x$$

Advanced Notation Note:
Rather than h for the small change in x, the formal notation is δx. So actually:

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

So we in fact have 3 symbols for "change in"!

- Δx: any change in x (as seen in Chp5: $m = \frac{\Delta y}{\Delta x}$)
- δx: a small change in x
- dx: an infinitesimally small change in x

So the estimated gradient using some point close by was $\frac{\delta y}{\delta x}$, but in the 'limit' as $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$

Example

The point A with coordinates $(4,16)$ lies on the curve with equation $y = x^2$.
At point A the curve has gradient g.

a) Show that $g = \lim_{h \to 0}(8 + h)$

b) Deduce the value of g.

a

?

b

?

Example

Prove from first principles that the derivative of $x^3 - 2x = 3x^2 - 2$

Test Your Understanding

Prove **from first principles** that the derivative of x^4 is $4x^3$.

?

Helping Hand:
"Row 4" of Pascal's Triangle is:
1 4 6 4 1
You're welcome.

Exercise 12B

Pearson Pure Mathematics Year 1/AS
Pages 261-262

(Note that Exercise 12A was skipped in these slides)

Don’t forget the ‘Challenge’ question on Page 262!

Just for your interest...

Why couldn't we just immediately make h equal to 0 in $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$?

If we just stick $h=0$ in straight away:

$$\lim_{h\to 0} \frac{(x+0)^2-x^2}{0} = \lim_{h\to 0} \frac{0}{0}$$

Wait, uh oh...

$\frac{0}{0}$ is known as an **indeterminate form**.
Whereas we know what happens with expressions like $\frac{1}{0}$ (i.e. its value tends towards infinity), indeterminate forms are bad because their values are ambiguous, and prevent lim expressions from being evaluated.

Consider $\frac{0}{0}$ for example: 0 divided by anything usually gives 0, but anything divided by 0 is usually infinity. We can see these conflict.

Can you guess some other indeterminate forms, i.e. expressions whose value is ambiguous?

?

Thankfully, when indeterminate forms appear in lim expressions, there are variety of techniques to turn the expression into one that doesn't have any indeterminate forms.

One simple technique, that worked in our example, is to expand and simplify. This gave us $\lim_{h\to 0}(2x+h)$ and $2x+0$ clearly is fine. Other techniques are more advanced. This is important when sketching harder functions:
http://www.drfrostmaths.com/resources/resource.php?rid=163

ME-WOW!

Differentiating x^n

Thankfully, there's a quick way to differentiate terms of the form x^n (where n is a constant) with having to use first principles every time:

✎ If $y = ax^n$ then $\frac{dy}{dx} = nax^{n-1}$ (where a, n are constants)
i.e. multiply by the power and reduce the power by 1

Examples:

$$y = x^5 \quad \rightarrow \quad \frac{dy}{dx} = \boxed{?}$$

Power is 5, so multiply by 5 then reduce power by 1.

$$f(x) = x^{\frac{1}{2}} \quad \rightarrow \quad \boxed{?}$$

The power need not be an integer! Remember to use $f'(x)$ not $\frac{dy}{dx}$

$$y = 2x^6 \quad \rightarrow \quad \boxed{?}$$

$$f(x) = \frac{x}{x^4} = \boxed{?} \quad \rightarrow \quad \boxed{?}$$

$$y = \sqrt{x^6} = \boxed{?} \rightarrow \quad \boxed{?}$$

Why would it be incorrect to say that $y = 2^x$ differentiates to $\frac{dy}{dx} = x\,2^{x-1}$?

?

Test Your Understanding

1 $y = x^7 \rightarrow \frac{dy}{dx} = \boxed{?}$

2 $y = 3x^{10} \rightarrow \boxed{?}$

3 $f(x) = \frac{x^{\frac{1}{2}}}{x^2} = \boxed{?} \rightarrow \boxed{?}$

4 $y = ax^a \rightarrow \boxed{?}$

5 $f(x) = \sqrt{49x^7} = \boxed{?} \rightarrow f'(x) = \boxed{?}$

Differentiating Multiple Terms

Differentiate $y = x^2 + 4x + 3$

$$\frac{dy}{dx} = \boxed{?}\,\boxed{?}\,\boxed{?}$$

First thing to note:
If $y = f(x) + g(x)$ then $\frac{dy}{dx} = f'(x) + g'(x)$
i.e. **differentiate each term individually in a sum/subtraction**.

$$y = 4x = 4x^1$$

Therefore applying the usual rule:

$$\frac{dy}{dx} = 4x^0 = 4$$

Alternatively, if you compare $y = 4x$ to $y = mx + c$, it's clear that the gradient is fixed and $m = 4$.

$$y = 3 = 3x^0$$

Therefore applying the usual rule:

$$\frac{dy}{dx} = 0x^{-1} = 0$$

Alternatively, if you sketch $y = 4$, the line is horizontal, so the gradient is 0.

Quickfire Questions

1. $y = 2x^2 - 3x \quad \rightarrow \quad \dfrac{dy}{dx} =$?

2. $y = 4 - 9x^3 \quad \rightarrow \quad \dfrac{dy}{dx} =$?

3. $y = 5x + 1 \quad \rightarrow \quad \dfrac{dy}{dx} =$?

4. $y = ax \quad \rightarrow \quad \dfrac{dy}{dx} =$?
   (where a is a constant)

5. $y = 6x - 3 + px^2 \quad \rightarrow \quad \dfrac{dy}{dx} =$?
   (where p is a constant)

Harder Example

Let $f(x) = 4x^2 - 8x + 3$

a) Find the gradient of $y = f(x)$ at the point $\left(\frac{1}{2}, 0\right)$

b) Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 8.

c) Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 4x - 5$.

a

?

Remember that the 'gradient function' allows you to find the gradient for a particular value of x.

b

?

This example is important!
Previously you used a value of x to get the gradient $f'(x)$.
This time we're **doing the opposite**: using a known gradient $f'(x)$ to get the value of x. We therefore substitute $f'(x)$ for 8.

Once you have your x, you need to work out y.
Ensure you use the correct equation!

c

?

Test Your Understanding

Let $f(x) = x^2 - 4x + 2$

a) Find the gradient of $y = f(x)$ at the point $(1, -1)$

b) Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 5.

c) Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 2 - x$.

a

?

b

?

c

?

Exercise 12D

Pearson Pure Mathematics Year 1/AS
Pages 265-266

(Note that Exercise 12C was skipped in these slides)

Differentiating Harder Expressions

If your expression isn't a sum of x^n terms, simply manipulate it until it is!

1. Turn roots into powers:

$y = \sqrt{x} = \boxed{?} \rightarrow \frac{dy}{dx} = \boxed{?}$

$y = \frac{1}{\sqrt[3]{x}} = \boxed{?} \rightarrow \frac{dy}{dx} = \boxed{?}$

2. Split up fractions.

$y = \frac{x^2 + 3}{\sqrt{x}} = \boxed{?}$

$\rightarrow \frac{dy}{dx} = \boxed{?}$

3. Expand out brackets.

$y = x^2(x - 3) = \boxed{?}$

$\rightarrow \frac{dy}{dx} = \boxed{?}$

4. Beware of numbers in denominators!

NOT $3x^{-1}$!!!

$y = \frac{1}{3x} = \boxed{?} \rightarrow \frac{dy}{dx} = \boxed{?}$

Test Your Understanding

Differentiate the following.

$$y = \frac{1}{\sqrt{x}} =$$?

$$y = \frac{2 + x^3}{x^2} =$$?

$$y = \frac{1 + 2x}{3x\sqrt{x}} =$$?

Exercise 12E

Pearson Pure Mathematics Year 1/AS
Pages 267-268

Extension

This just means "differentiate twice". We'll be looking at the 'second derivative' later in this chapter.

[MAT 2013 1E]

The expression $\frac{d^2}{dx^2}[(2x-1)^4(1-x)^5]+\frac{d}{dx}[(2x+1)^4(3x^2-2)^2]$ is a polynomial of degree:

A) 9
B) 8
C) 7
D) less than 7

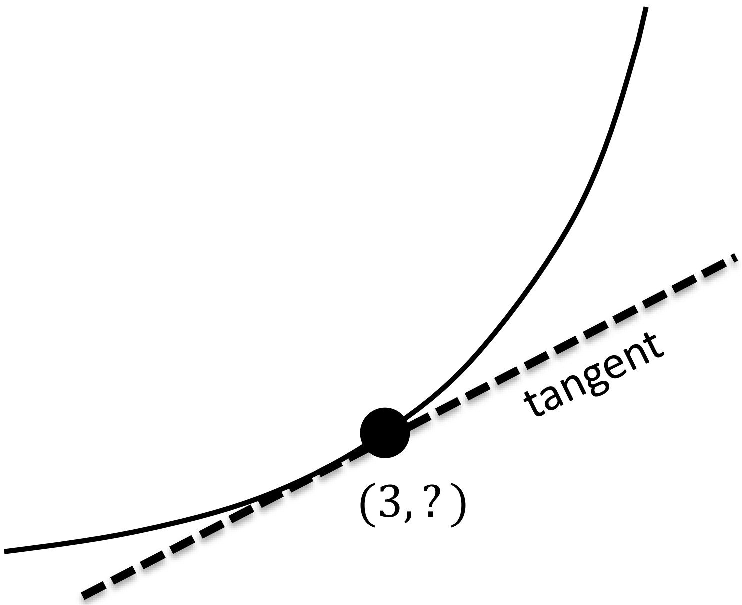
?

Finding equations of tangents

Find the equation of the **tangent** to the curve $y = x^2$ when $x = 3$.

We want to use $y - y_1 = m(x - x_1)$ for the tangent (as it is a straight line!).
Therefore we need:

(a) A point (x_1, y_1)
(b) The gradient m.



Gradient function: ?

Gradient when $x = 3$: ?

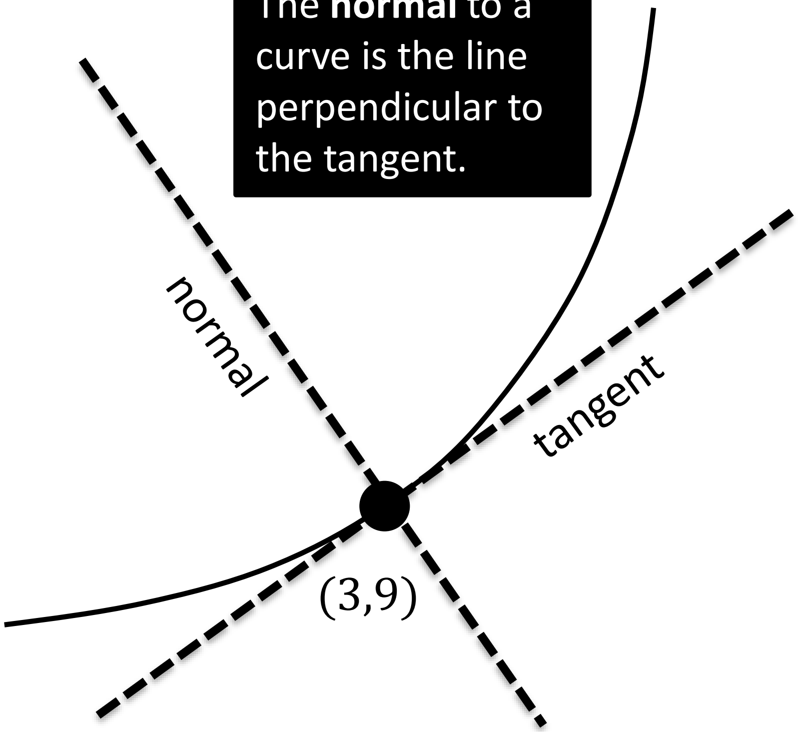
y-value when $x = 3$: ?

So equation of tangent: ?

Finding equations of normals

Find the equation of the **normal** to the curve $y = x^2$ when $x = 3$.

The **normal** to a curve is the line perpendicular to the tangent.



Equation of tangent (from earlier):

$$\boldsymbol{y - 9 = 6(x - 3)}$$

Therefore equation of normal:

$$\boldsymbol{y - 9 =} \text{ ?}$$

Fro Exam Tip: A **very common error** is for students to accidentally forget whether the question is asking for the tangent or for the normal.

Test Your Understanding

Find the equation of the **normal** to the curve $y = x + 3\sqrt{x}$ when $x = 9$.

When $x = 9$, $y =$? y

? gradient

Fro Tip: I like to use m_T and m_N to make clear to the examiner (and myself) what gradient I've found.

? Final equation

Exercise 12F

Pearson Pure Mathematics Year 1/AS
Pages 269-270

The 'difference of two cubes':
$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
will help for both of these.

Extension

1 *[STEP I 2005 Q2]*
The point P has coordinates $(p^2, 2p)$ and the point Q has coordinates $(q^2, 2q)$, where p and q are non-zero and $p \neq q$. The curve C is given by $y^2 = 4x$. The point R is the intersection of the tangent to C at P and the tangent to C at Q. Show that R has coordinates $(pq, p+q)$.

The point S is the intersection of the normal to C at P and the normal to C at Q. If p and q are such that $(1,0)$ lies on the line PQ, show that S has coordinates $(p^2 + q^2 + 1, p + q)$, and that the quadrilateral $PSQR$ is a rectangle.

Solutions on next slide.

2 *[STEP I 2012 Q4]*
The curve C has equation $xy = \frac{1}{2}$.
The tangents to C at the distinct points $P\left(p, \frac{1}{2p}\right)$ and $Q\left(q, \frac{1}{2q}\right)$, where p and q are positive, intersect at T and the normal to C at these points intersect at N. Show that T is the point

$$\left(\frac{2pq}{p+q}, \frac{1}{p+q}\right)$$

In the case $pq = \frac{1}{2}$, find the coordinates of N. Show (in this case) that T and N lie on the line $y = x$ and are such that the product of their distances from the origin is constant.

Solutions to Extension Question 1

$y^2 = 4x \Rightarrow 2yy' = 4 \Rightarrow y' = \dfrac{2}{y}$

This is using something called 'implicit differentiation' (Year 2), but you could easily do:

$$y^2 = 4x \rightarrow y = 2x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}} = \frac{2}{y}$$

Equation of tangent at P: $y - 2p = \dfrac{1}{p}\left(x - p^2\right) \Rightarrow py = x + p^2$.

Equation of tangent at Q: $qy = x + q^2$

Intersect where $qy - q^2 = py - p^2$

$\Rightarrow (q-p)\,y = q^2 - p^2$

$\Rightarrow y = p + q \Rightarrow x = pq$ by substitution. Hence $R\,(pq\,,\; p+q)$.

Equation of normal at P: $y - 2p = -p\left(x - p^2\right) \Rightarrow y + px = 2p + p^3$.

Equation of normal at Q: $y + qx = 2q + q^3$

Intersect where $2p + p^3 - px = 2q + q^3 - qx$

$\Rightarrow x\,(p-q) = 2\,(p-q) + (p-q)\left(p^2 + pq + q^2\right)$ using the identity $p^3 - q^3 \equiv (p-q)\left(p^2+pq+q^2\right)$

$\Rightarrow x = p^2 + pq + q^2 + 2 \Rightarrow y = -pq\,(p+q)$ by substitution.

But $(1\,,\;0)$ lies on PQ so the gradient of the line segement from P to $(1\,,\;0)$ equals the gradient of the line segment from Q to $(1\,,\;0)$.

$\Rightarrow \dfrac{2p}{p^2-1} = \dfrac{2q}{q^2-1}$

$\Rightarrow 2pq^2 - 2p = 2qp^2 - 2q \Rightarrow pq\,(q-p) = p - q \Rightarrow pq = -1$

$\Rightarrow S$, where the normals intersect, has coordinates $\left(p^2 + q^2 + 1\,,\; p+q\right)$.

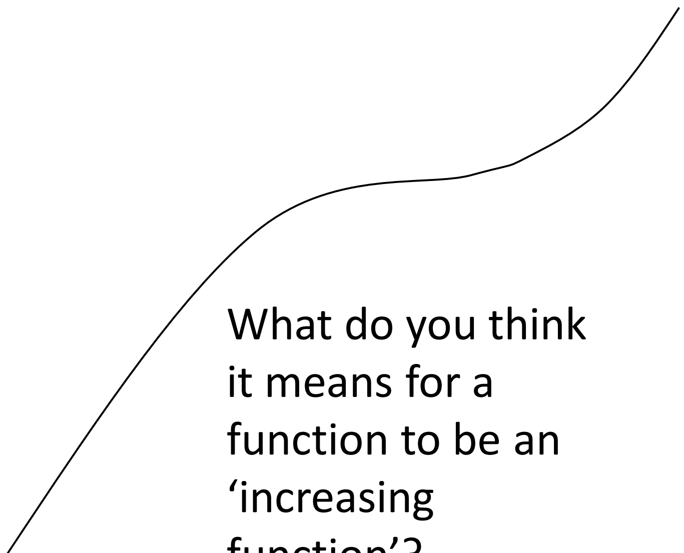
Obviously, SP is perpendicular to PR and QS is perpendicular to QR, because each of these is a tangent-normal pair.

Furthermore, the gradient of PR $\times$ the gradient of QR $= \frac{1}{p} \times \frac{1}{q} = \frac{1}{pq} = -1$, so PR is perpendicular to QR.

Also, the gradient of PS $\times$ the gradient of $QS = -p \times -q = pq = -1$, so PS is perpendicular to QS.

Therefore all four angles are right angles, proving that PSQR is a rectangle. It is not sufficient to consider only the lengths of the sides, since the quadrilateral could be a parallelogram.

Increasing and Decreasing Functions

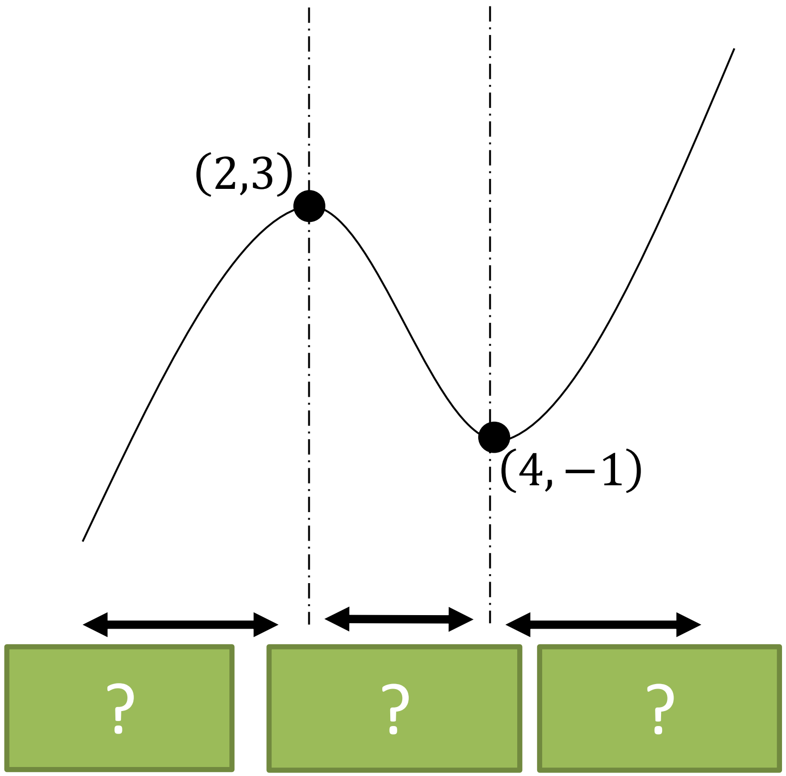


What do you think it means for a function to be an 'increasing function'?

✏ An increasing function is one whose gradient is always at least 0.
$f'(x) \geq 0$ for all x.

It would be '**strictly increasing**' if $f(x) > 0$ for all x, i.e. is not allowed to go horizontal.

A function can also be increasing and decreasing in certain intervals.



We could also write "$f(x)$ is decreasing in the interval $[2,4]$"
$[a, b]$ **represents all the real numbers between** a **and** b **inclusive, i.e:**

$$[a, b] = \{x : a \leq x \leq b\}$$

Examples

Show that the function $f(x) = x^3 + 6x^2 + 21x + 2$ is increasing for all real values of x.

?

Find the interval on which the function $f(x) = x^3 + 3x^2 - 9x$ is decreasing.

?

Tip: To show a quadratic is always positive, complete the square, then indicate the squared term is always at least 0.

Test Your Understanding

Show that the function $f(x) = x^3 + 16x - 2$ is increasing for all real values of x.

?

Find the interval on which the function $f(x) = x^3 + 6x^2 - 135x$ is decreasing.

?

Exercise 12G

Pearson Pure Mathematics Year 1/AS
Page 271

Second Order Derivatives

When you differentiate once, the expression you get is known as the **first derivative**. Unsurprisingly, when we differentiate a second time, the resulting expression is known as the **second derivative**. And so on…

Original Function	First Derivative	Second Derivative
$y = x^4$	Leibniz's: $\dfrac{dy}{dx} = 4x^3$	$\dfrac{d^2y}{dx^2} = 12x^2$
	$y' = 4x^3$	$y'' = 12x^2$
	Newton's: $\dot{y} = 4x^3$	$\ddot{y} = 12x^2$

Lagrange's

$$f(x) = x^4 \longrightarrow f'(x) = 4x^3 \longrightarrow f''(x) = 12x^2$$

You can similarly have the third derivative ($\frac{d^3y}{dx^3}$), although this is no longer in the A Level syllabus. We'll see why might use the second derivative soon…

Just for your interest...

How does the notation $\frac{d^2y}{dx^2}$ work?
Why are the squareds where they are?

Suppose that $y = x^3 + 1$.
Then when we write $\frac{dy}{dx}$, we're effectively doing $\frac{d(x^3+1)}{dx}$ (by substitution), although this would typically be written:

$$\frac{d}{dx}(x^3 + 1)$$

The $\frac{d}{dx}(\ldots)$ notation is quite handy, because it behaves as a function and allows us to write the original expression and the derivative within a single equation:

$$\frac{d}{dx}(x^3 + 1) = 3x^2$$

Therefore, if we wanted to differentiate y twice, we'd do:

$$\frac{d}{dx}\left(\frac{d}{dx}(y)\right)$$
$$= \frac{d^2}{dx^2}(y)$$
$$= \frac{d^2y}{dx^2}$$

ME-WOW!

Examples

If $y = 3x^5 + \frac{4}{x^2}$, find $\frac{d^2y}{dx^2}$.

?

If $f(x) = 3\sqrt{x} + \frac{1}{2\sqrt{x}}$, find $f''(x)$.

?

Test Your Understanding

If $y = 5x^3 - \frac{x}{3\sqrt{x}}$, find $\frac{d^2y}{dx^2}$.

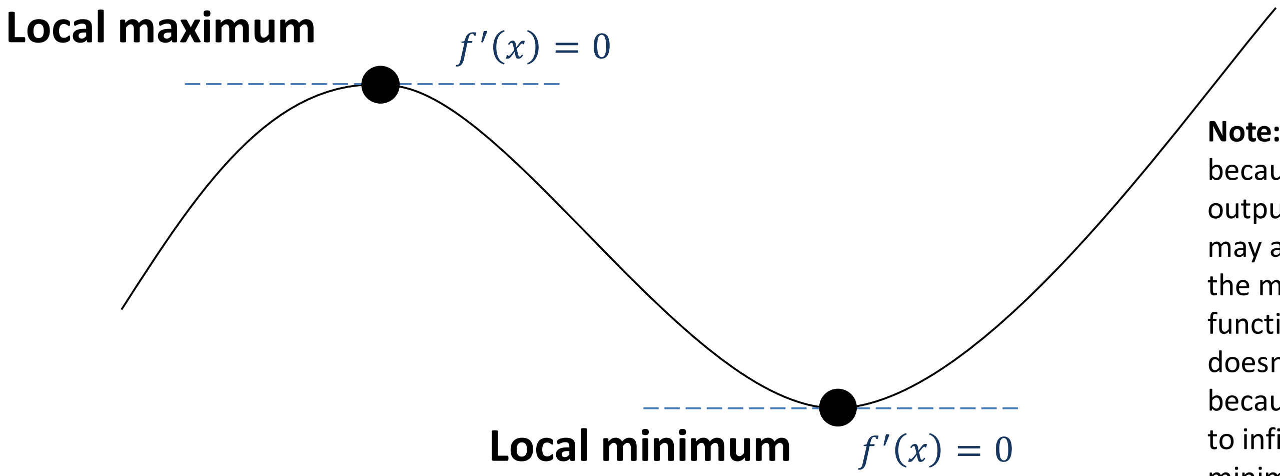
?

(Note: For time reasons, I'm skipping Exercise 12H on Page 272)

Also note an error on this page: "*When you differentiate with respect to x, you treat any other letters as constants*."
This is emphatically not true – **it depends on whether the letter is a variable or a constant**.
The statement however is true of 'partial differentiation' (which is not in the A Level syllabus). In fact in Year 2, you will learn that xy, when differentiated with respect to x, gives $x\frac{dy}{dx} + y$, not to just y. What the textbook means is "*When you differentiate with respect to x, you treat any letters, defined to be constants, as numbers*." So ax would differentiate to a, just as $3x$ differentiates to 3, but ONLY if you were told that a is a constant! If a was in fact a variable, we need to use something called the *product rule* (Year 2 content).

Stationary/Turning Points

A stationary point is where the gradient is 0, i.e. $f'(x) = 0$.



Note: It's called a '**local**' maximum because it's the function's largest output within the vicinity. Functions may also have a '**global**' maximum, i.e. the maximum output across the entire function. This particular function doesn't have a global maximum because the output keeps increasing up to infinity. It similarly has no global minimum, as with all cubics.

Find the coordinates of the turning points of $y = x^3 + 6x^2 - 135x$

?

More Examples

Find the least value of
$$f(x) = x^2 - 4x + 9$$

?

Method 1:
Differentiation

?

Method 2:
Completing the Square

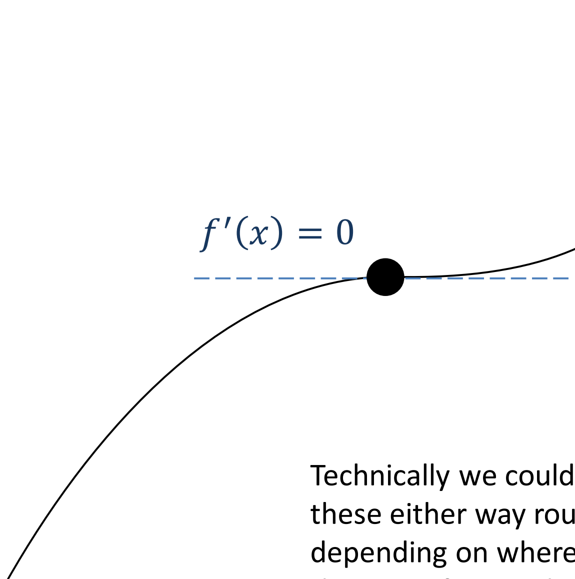
Fro Note: Method 2 is only applicable for quadratic functions. For others, differentiation must be used.

Find the turning point of
$$y = \sqrt{x} - x$$

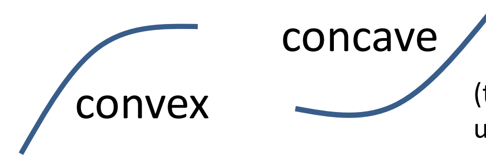
?

Points of Inflection

There’s a third type of stationary point (that we’ve encountered previously):



A point of inflection is where the curve changes from convex to concave (or vice versa).

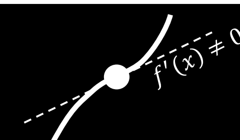


(the same terms used in optics!)

Technically we could label these either way round depending on where we view the curve from. What’s important is that the **concavity changes**.

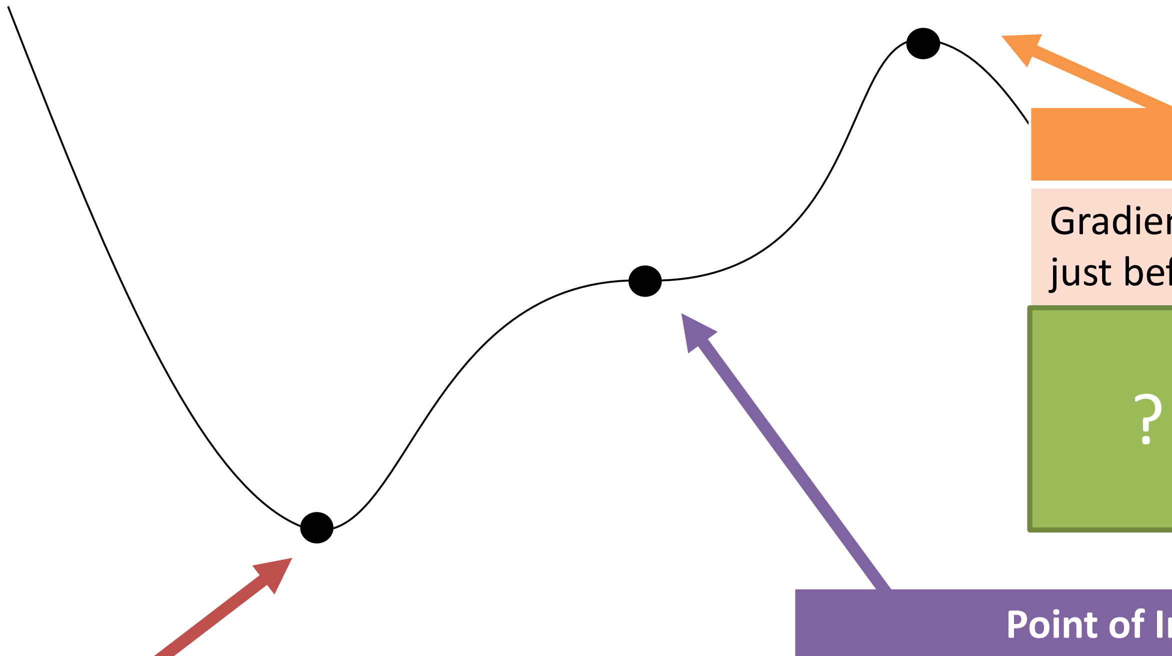
i.e. the line curves in one direction before the point of inflection, then curves in the other direction after.

Side Note: Not all points of inflection are stationary points, as can be seen in the example on the right.
A point of inflection which **is** a stationary point is known as a *saddle point*.



How do we tell what type of stationary point?

Method 1: Look at gradient just before and just after point.



Local Minimum		
Gradient just before	Gradient at minimum	Gradient just after
?	?	?

Point of Inflection		
Gradient just before	Gradient at p.o.i	Gradient just after
?	?	?

Local Maximum		
Gradient just before	Gradient at maximum	Gradient just after
?	?	?

How do we tell what type of stationary point?

Method 1: Look at gradient just before and just after point.

Find the stationary point on the curve with equation $y = x^4 - 32x$, and determine whether it is a local maximum, a local minimum or a point of inflection.

? Turning Point

Strategy: ?

? Determine point type

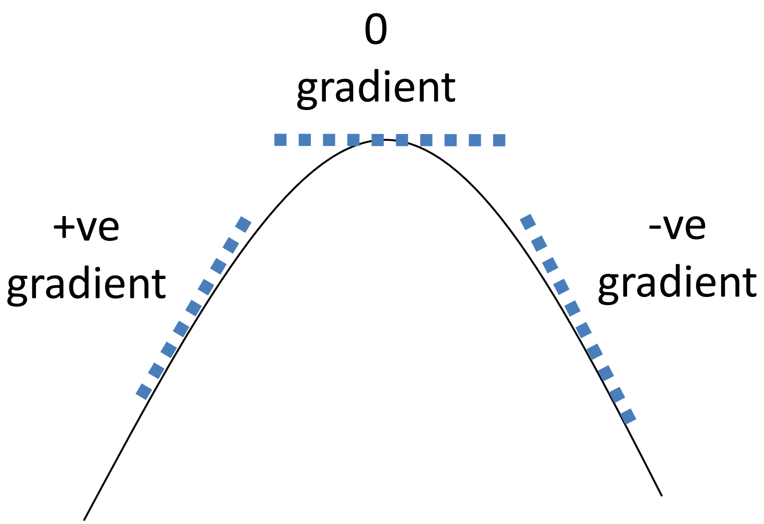
Method 2: Using the second derivative

The method of substituting values of x just before and after is a bit cumbersome.
It also has the potential for problems: what if two different types of stationary points are really close together?

Recall the gradient gives a measure of the **rate of change** of y, i.e. how much the y value changes as x changes.

Thus by differentiating the gradient function, the **second derivative tells us the rate at what the gradient is changing**.

Thus if the second derivative is positive, the gradient is increasing.
If the second derivative is negative, the gradient is decreasing.



At a maximum point, we can see that as x increases, the gradient is decreasing from a positive value to a negative value.

$$\therefore \frac{d^2y}{dx^2} < 0$$

Method 2: Using the second derivative

At a stationary point $x = a$:
- If $f''(a) > 0$ the point is a local minimum.
- If $f''(a) < 0$ the point is a local maximum.
- If $f''(a) = 0$ it could be any type of point, so resort to Method 1.

I will eventually do a 'Just for your Interest…' thingy on why we can't classify the point when $f''(x) = 0$, and how we could use the **third derivative**!

The stationary point of $y = x^4 - 32x$ is $(2, -48)$. Use the second derivative to classify this stationary point.

?

Test Your Understanding

Edexcel C2 May 2013 Q9

The curve with equation

$$y = x^2 - 32\sqrt{x} + 20, \qquad x > 0,$$

has a stationary point P.

Use calculus

(a) to find the coordinates of P,

(6)

(b) to determine the nature of the stationary point P.

(3)

(a)

?

(b)

?

Sketching Graphs

All the way back in Chapter 4, we used features such as intercepts with the axes, and behaviour when $x \to \infty$ and $x \to -\infty$ in order to sketch graphs.
Now we can also find stationary/turning points!

[Textbook] By first finding the stationary points, sketch the graph of $y = \frac{1}{x} + 27x^3$

? Turning Points

? As $x \to \infty$, $x \to -\infty$

? Vertical Asymptotes

? Graph

Exercise 12I

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Extension

1 [MAT 2014 1C] The cubic
$$y = kx^3 - (k+1)x^2 + (2-k)x - k$$
has a turning point, that is a minimum, when $x = 1$ precisely for

A) $k > 0$
B) $0 < k < 1$
C) $k > \frac{1}{2}$
D) $k < 3$
E) all values of k

2 *[MAT 2004 1B]* The smallest value of the function:
$$f(x) = 2x^3 - 9x^2 + 12x + 3$$
In the range $0 \leq x \leq 2$ is what?

3 *[MAT 2001 1E]* The maximum gradient of the curve $y = x^4 - 4x^3 + 4x^2 + 2$ in the range $0 \leq x \leq 2\frac{1}{5}$ occurs when:

A) $x = 0$
B) $x = 1 - \frac{1}{\sqrt{3}}$
C) $x = 1 + \frac{1}{\sqrt{3}}$
D) $x = 2\frac{1}{5}$

Hint: When two curves touch, their y values must match, but what else must also match?

4 [STEP I 2007 Q8] A curve is given by:
$$y = ax^3 - 6ax^2 + (12a + 12)x - (8a + 16)$$
where a is a real number. Show that this curve touches the curve with equation $y = x^3$ at $(2,8)$. Determine the coordinates of any other point of intersection of the two curves.

(i) Sketch on the same axes these two curves when $a = 2$.

(ii) … when $a = 1$ (iii) when $a = -2$

Solutions to Extension Questions

1 [MAT 2014 1C] The cubic

$$y = kx^3 - (k+1)x^2 + (2-k)x - k$$

has a turning point, that is a minimum, when $x = 1$ precisely for

A) $k > 0$
B) $0 < k < 1$
C) $k > \frac{1}{2}$
D) $k < 3$
E) all values of k

$$\frac{dy}{dx} = 3kx^2 - 2(k+1)x + (2-k)$$

When $x = 1$, $\frac{dy}{dx} = 3k - 2k - 2 + 2 - k \equiv 0$

Therefore there is a turning point for all values of k. However, this must be a minimum.

$$\frac{d^2y}{dx^2} = 6kx - 2(k+1)$$

When $x = 1$, $\frac{d^2y}{dx^2} = 4k - 2$

If minimum: $4k - 2 > 0 \quad \rightarrow \quad k > \frac{1}{2}$

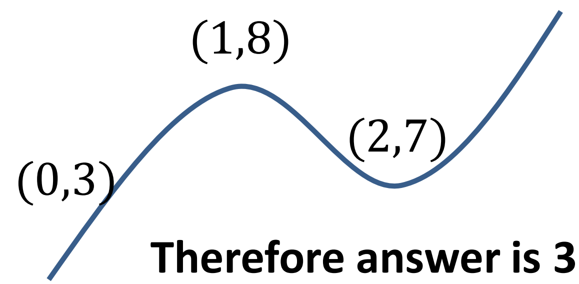
2 *[MAT 2004 1B]* The smallest value of the function:

$$f(x) = 2x^3 - 9x^2 + 12x + 3$$

In the range $0 \leq x \leq 2$ is what?

$$f'(x) = 6x^2 - 18x + 12 = 0$$
$$x^2 - 3x + 2 = 0$$
$$(x-1)(x-2) = 0$$
$$x = 1 \quad \rightarrow \quad f(1) = 8$$
$$x = 2 \quad \rightarrow \quad f(2) = 7$$

At start of range: $f(0) = 3$



Therefore answer is 3.

Solutions to Extension Questions

3 [MAT 2001 1E] The maximum gradient of the curve $y = x^4 - 4x^3 + 4x^2 + 2$ in the range $0 \leq x \leq 2\frac{1}{5}$ occurs when:

A) $x = 0$

B) $x = 1 - \frac{1}{\sqrt{3}}$

C) $x = 1 + \frac{1}{\sqrt{3}}$

D) $x = 2\frac{1}{5}$

Gradient: $\boldsymbol{\frac{dy}{dx} = 4x^3 - 12x^2 + 8x}$

So find max value of $\boldsymbol{4x^3 - 12x^2 + 8x}$

$$\frac{d^2y}{dx^2} = 12x^2 - 24x + 8 = 0$$

$$3x^2 - 6x + 2 = 0$$

$$x = 1 \pm \frac{1}{\sqrt{3}}$$

Due to the shape of a cubic, we have the a local maximum gradient at $\boldsymbol{x = 1 - \frac{1}{\sqrt{3}}}$ **and a local minimum at** $\boldsymbol{x = 1 + \frac{1}{\sqrt{3}}}$

In range $\boldsymbol{0 \leq x \leq 2\frac{1}{5}}$**, answer must either be** $\boldsymbol{x = 1 + \frac{1}{\sqrt{3}}}$ **or** $\boldsymbol{x = 2\frac{1}{5}}$**. Substituting these into** $\boldsymbol{y}$ **yields a higher value for the latter, so answer is (D).**

Solutions to Extension Questions

4 [STEP I 2007 Q8] A curve is given by:

$y = ax^3 - 6ax^2$
$+(12a + 12)x - (8a + 16)$

where a is a real number. Show that this curve touches the curve with equation $y = x^3$ at (2,8). Determine the coordinates of any other point of intersection of the two curves.

(i) Sketch on the same axes these two curves when $a = 2$.
(ii) … when $a = 1$
(iii) … when $a = -2$

Consider $\mathrm{f}(x) = ax^3 - 6ax^2 + (12a + 12)x - (8a + 16)$.

Since $\mathrm{f}(2) = 8$ and $\mathrm{f}'(2) = 12$, the curve $y = \mathrm{f}(x)$ touches $y = x^3$ at $(2, 8)$: notice that both calculations are necessary to prove that the curves **touch**.

To find the other intersection point, let $\mathrm{f}(x) = x^3$

$\Rightarrow (a-1)x^3 - 6ax^2 + (12a + 12)x - (8a + 16) = 0$

Substituting $x = 2$: $8(a-1) - 24a + 2(12a + 12) - (8a + 16) \equiv 0$

$\Rightarrow (x-2)\left[(a-1)x^2 - (4a+2)x + (4a+8)\right] = 0$ (notice that factorising by inspection is much easier than using a method such as long division)

Substituting $x = 2$ into the quadratic factor: $4(a-1) - 2(4a+2) + (4a+8) \equiv 0$.

$\Rightarrow (x-2)(x-2)\left[(a-1)x - (2a+4)\right] = 0$

So the other intersection point has coordinates $\left(\dfrac{2a+4}{a-1}, \left[\dfrac{2a+4}{a-1}\right]^3\right)$

(i) When $a = 2$, $\dfrac{2a+4}{a-1} = 8$.

Hence the two graphs touch at $(2, 8)$ and intersect at $(8, 512)$.

$y = 2x^3 - 12x^2 + 36x - 32$ has no turning points: consider the derivative $6x^2 - 24x + 36 = 0$.

(ii) When $a = 1$, $\dfrac{2a+4}{a-1}$ is undefined.

Hence the two graphs touch at $(2, 8)$, and do not intersect elsewhere.

$y = x^3 - 6x^2 + 24x - 24$ has no turning points: consider the derivative $3x^2 - 12x + 24 = 0$.

(iii) When $a = -2$, $\dfrac{2a+4}{a-1} = 0$.

Hence the two graphs touch at $(2, 8)$ and intersect at $(0, 0)$.

$y = -2x^3 + 12x^2 - 12x$ turns when $x = 2 \pm \sqrt{2} \approx 3.4$ and 0.6: consider the derivative $-6x^2 + 24x - 12 = 0$.

Sketching Gradient Functions

You may be asked to sketch the gradient function. This means we have to draw the graph of $y = \frac{dy}{dx}$.

If you know the original function, you can find $f'(x)$ and you should be able to sketch y = $f'(x)$ using skills we have learned previously.

Example: Sketch the gradient function for the function $f(x) = x^2 + 3x + 2$

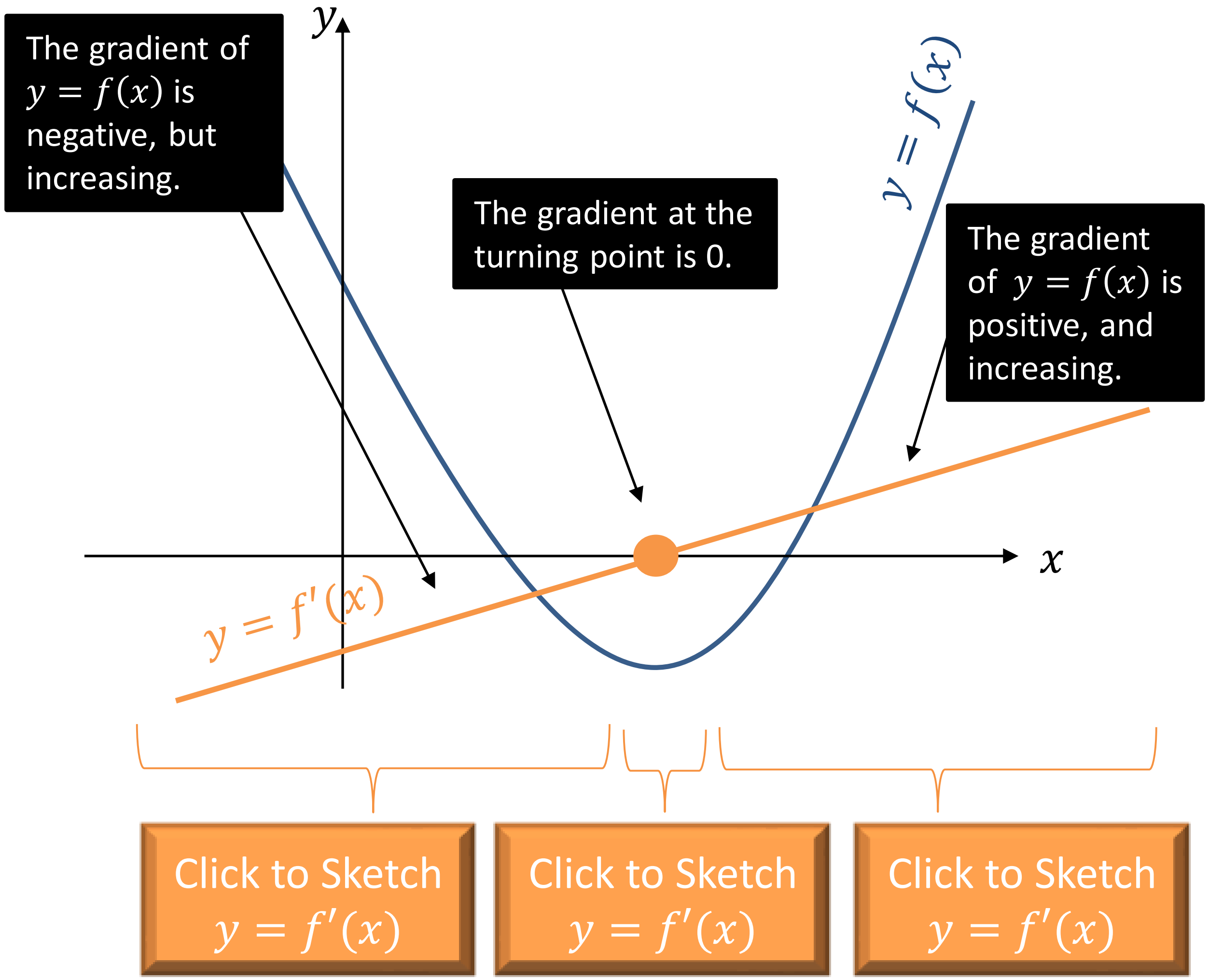
?

?

If the original function is a cubic we would expect a quadratic graph for $y = \frac{dy}{dx}$
If the original function is a quartic we would expect a cubic graph.

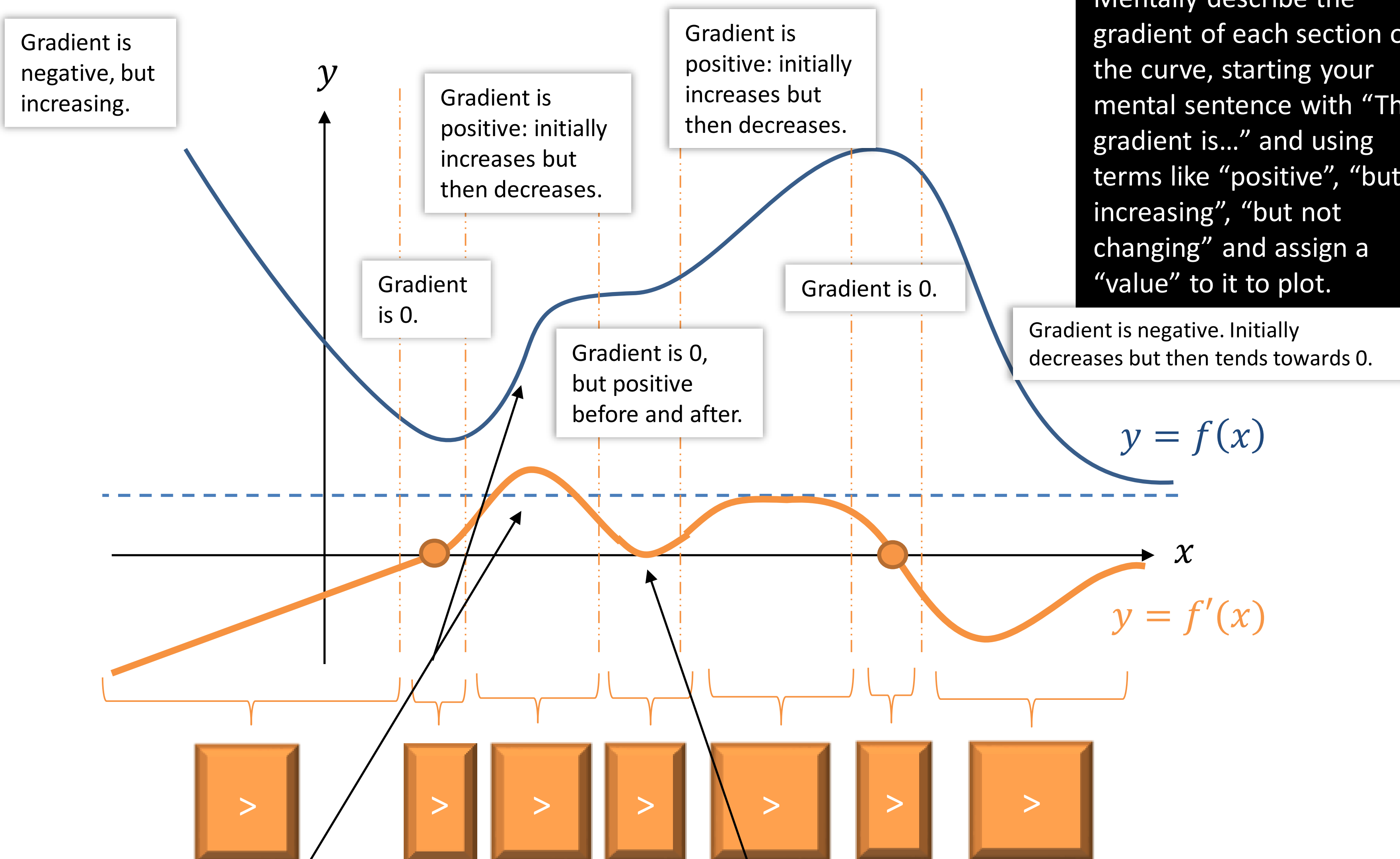
Sketching Gradient Functions

Sometimes **you won't be given the function explicitly**, you will only be given **the sketch**.



A Harder One

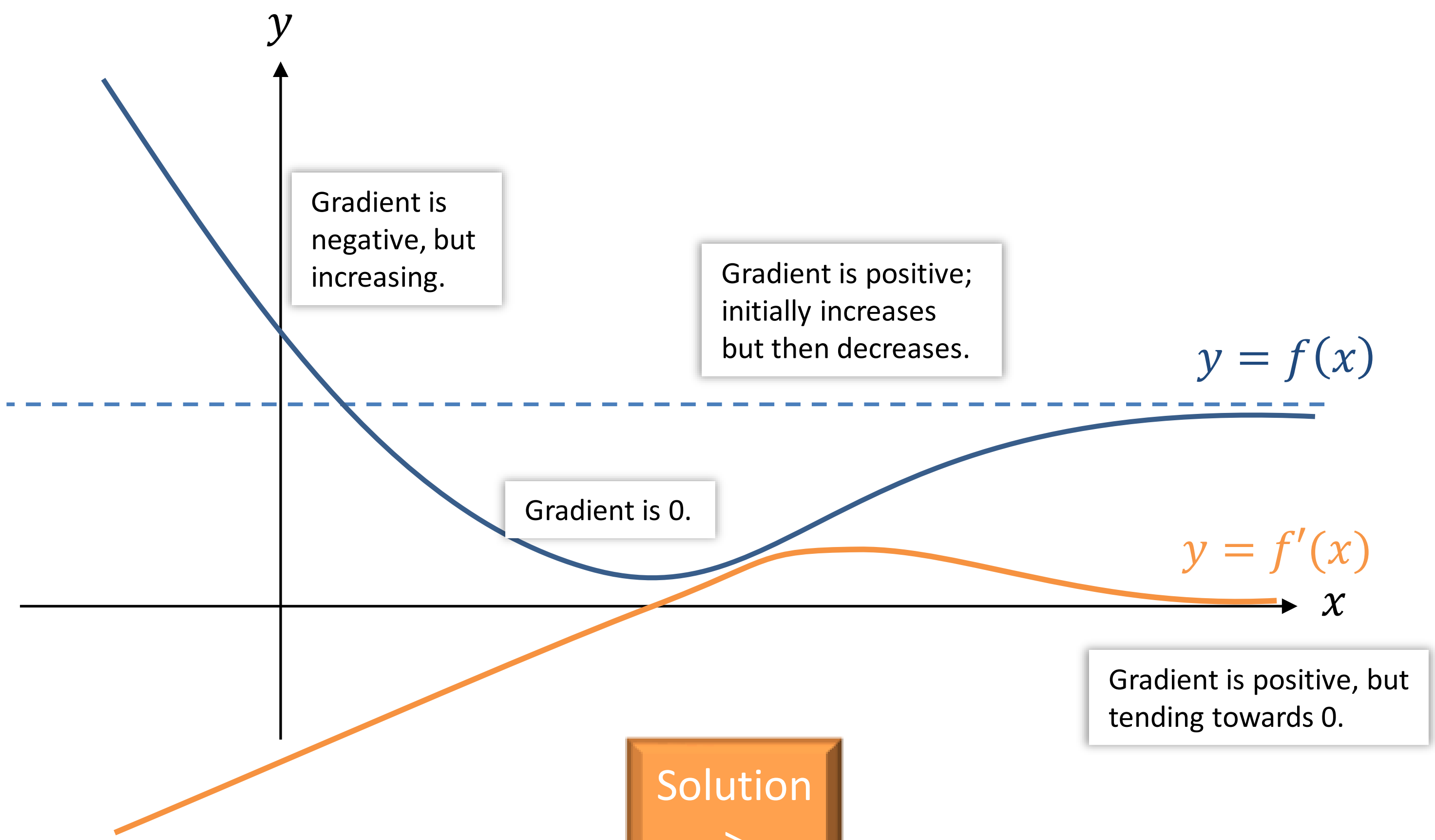
Mentally describe the gradient of each section of the curve, starting your mental sentence with “The gradient is…” and using terms like “positive”, “but increasing”, “but not changing” and assign a “value” to it to plot.



Fro Pro Tip: The gradient is momentarily not changing so the gradient of the gradient is 0. We get a **turning point** in $y = f'(x)$ whenever the **curve has a point of inflection**.

Again, on $y = f(x)$ this is a point of inflection so the gradient function has a turning point. It is also a stationary point, so $f'(x) = 0$. i.e. The $f'(x)$ curve touches the x-axis.

Test Your Understanding



Solution >

Summary

Y = f(x)	Y = f'(x)
max / min	Cuts the x - axis
Point of inflection	Touches the x – axis
Positive gradient	Above the x - axis
Negative gradient	Below the x - axis
Vertical asymptote	Vertical asymptote
Horizontal asymptote	Horizontal asymptote at x-axis

Exercise 12J

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Extension

1 [MAT 2015 1B]

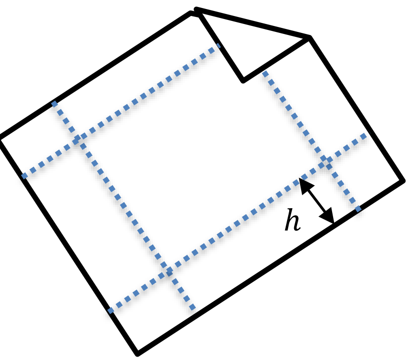
$$f(x) = (x + a)^n$$

where a is a real number and n is a positive whole number, and $n \geq 2$. If $y = f(x)$ and $y = f'(x)$ are plotted on the same axes, the number of intersections between $f(x)$ and $f'(x)$ will:

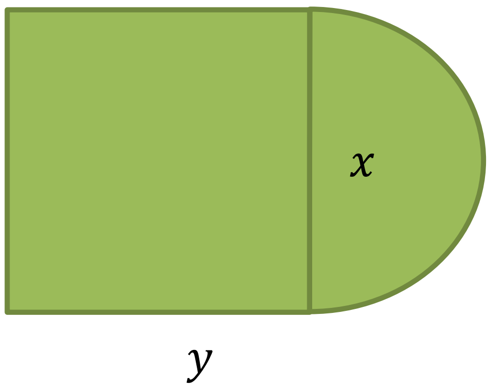
A) always be odd
B) always be even
C) depend on a but not n
D) depend on n but not a
E) depend on both a and n.

?

Optimisation Problems/Modelling



We have a sheet of A4 paper, which we want to fold into a cuboid. What height should we choose for the cuboid in order to maximise the volume?



We have 50m of fencing, and want to make a bear pen of the following shape, such that the area is maximised. What should we choose x and y to be?

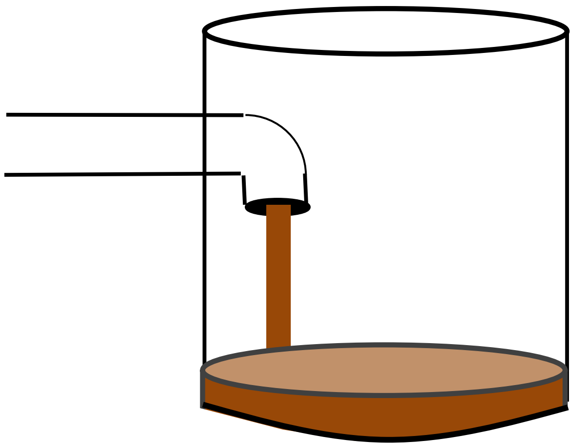
These are examples of **optimisation problems**: we're trying to maximise/minimise some quantity by choosing an appropriate value of a variable that we can control.

We can use differentiation to solve optimisation problems because it allows us to find maximum and minimum values of functions.

'Rate of change'

Up to now we've had y in terms of x, where $\frac{dy}{dx}$ means "the **rate** at which y changes with respect to x".

We can use similar gradient function notation for other **physical quantities**.



A sewage container fills at a rate of 20 cm^3 per second.

How could we use appropriate notation to represent this?

$$\frac{dV}{dt} = 20\ cm^3/s$$

"The rate at which the volume V changes with respect to time t."

Example Optimisation Problem

Optimisation problems in an exam usually follow the following pattern:

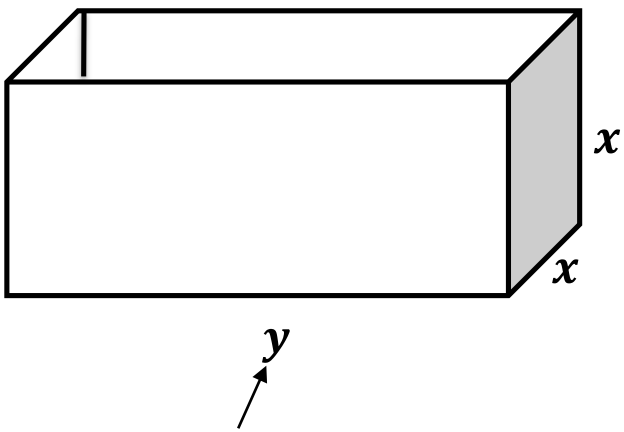
- There are 2 variables involved (you may have to introduce one yourself), typically lengths.
- There are expressions for **two different physical quantities**:
  - One is a **constraint**, e.g. "the surface area is $20cm^2$".
  - The **other we wish to maximise/minimise**, e.g. "we wish to maximise the volume".
- We use the constraint to **eliminate one of the variables** in the latter equation, so that it is then **just in terms of one variable**, and we can then use differentiation to find the turning point.

Example Optimisation Problem

A large tank in the shape of a cuboid is to be made from 54m^2 of sheet metal. The tank has a horizontal base and no top. The height of the tank is x metres. Two of the opposite vertical faces are squares.

a) Show that the volume, V m^3, of the tank is given by

$$V = 18x - \frac{2}{3}x^3.$$



We need to introduce a second variable ourselves so that we can find expressions for the surface area and volume.

What do we know?

?

These are the two equations mentioned in the guidance: one for surface area and one volume.

But we want V just in terms of x:

$y =$? $\rightarrow$ $V =$?

$V =$?

b) Given that x can vary, use differentiation to find the maximum or minimum value of V.

?

?

Once we have the 'optimal' value of x, we sub it back into V to get the best possible volume.

Test Your Understanding

Edexcel C2 May 2011 Q8

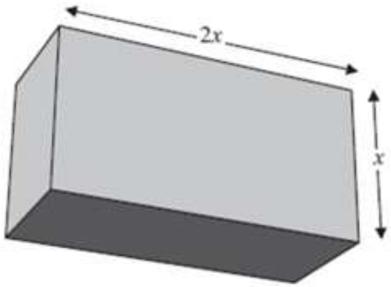


Figure 2

A cuboid has a rectangular cross-section where the length of the rectangle is equal to twice its width, x cm, as shown in Figure 2.

The volume of the cuboid is 81 cubic centimetres.

(*a*) Show that the total length, L cm, of the twelve edges of the cuboid is given by

$$L = 12x + \frac{162}{x^2}.$$

(3)

(*b*) Use calculus to find the minimum value of L.

(6)

(*c*) Justify, by further differentiation, that the value of L that you have found is a minimum.

(2)

(b)

? b

(c)

? c

(a)

? a

Exercise 12K

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Extension

1 [STEP I 2006 Q2] A small goat is tethered by a rope to a point at ground level on a side of a square barn which stands in a large horizontal field of grass. The sides of the barn are of length $2a$ and the rope is of length $4a$. Let A be the area of the grass that the goat can graze.
Prove that $A \leq 14\pi a^2$ and determine the minimum value of A.

?