P1 Chapter 4 :: Graphs & Transformations

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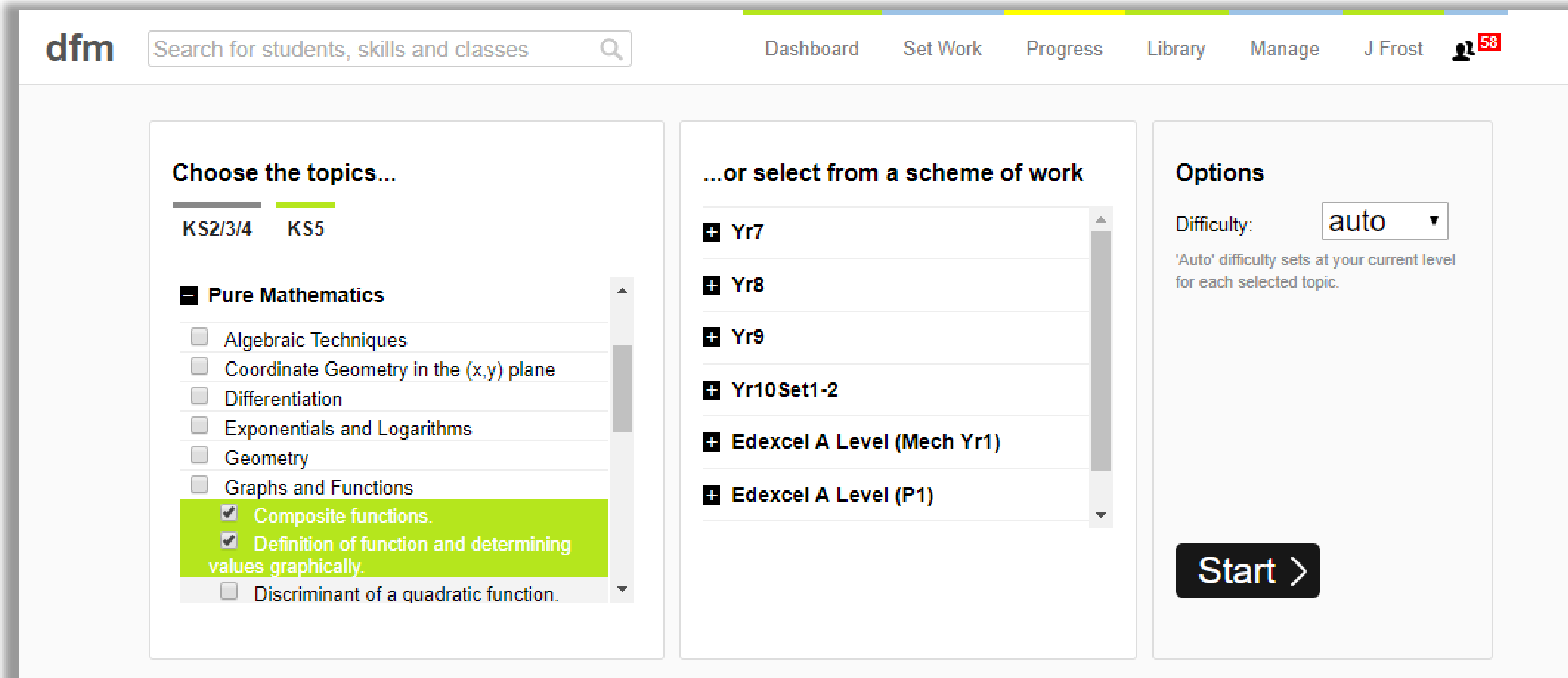

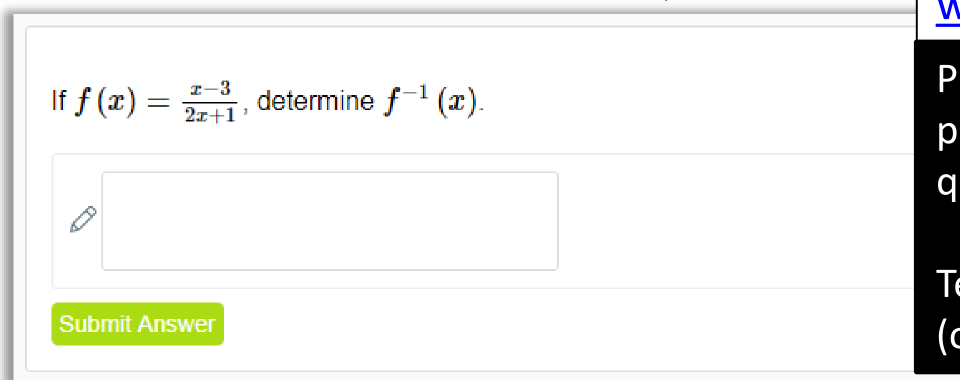

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Practise questions by chapter, including past paper Edexcel questions and extension questions (e.g. MAT).

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Chapter Overview

There are a few new bits and pieces since GCSE!

1a:: Cubic Graphs

Sketch the graph with equation:

$$y = x(x-3)^2$$

1b:: Quartic Graphs

Sketch the graph with equation:

$$y = (x-1)^2(x+1)^2$$

NEW! to A Level 2017+
The old A Level only included cubic graphs, not quartics.

1c:: Reciprocal Graphs

Sketch the graph with equation $y = -\frac{3}{x^2}$

NEW! to A Level 2017+
In addition to graphs of the form $y = \frac{a}{x}$, you now need to recognise the sketch of $y = \frac{a}{x^2}$

2:: Points of Intersection

Sketch the curves $y = \frac{4}{x^2}$ and $y = x^2(x-1)$ on the same axes. Using your sketch, state, with a reason, the number of real solutions to the equation $x^4(x-1) - 4 = 0$.

3:: Graph Transformations

If $f(x) = x^2(x+1)$, sketch the graph of $y = f(x+a)$, indicating any intercepts with the axes.

NEW! since GCSE
The GCSE 2015+ syllabus included translations of graphs but not stretches.

Polynomial Graphs

In Chapter 2 we briefly saw that a **polynomial** expression is of the form:

$$a + bx + cx^2 + dx^3 + ex^3 + \cdots$$

where a, b, c, d, e, … are constants (which could be 0).

The **order** of a polynomial is its highest power.

Order	Name
0	Constant (e.g. "4")
1	?
2	?
3	?
4	?
5	?

These are covered in Chapter 5.

Chapter 2 explored the graphs for these.

We will cover these now.

While these are technically beyond the A Level syllabus, we will look at how to sketch polynomials in general.

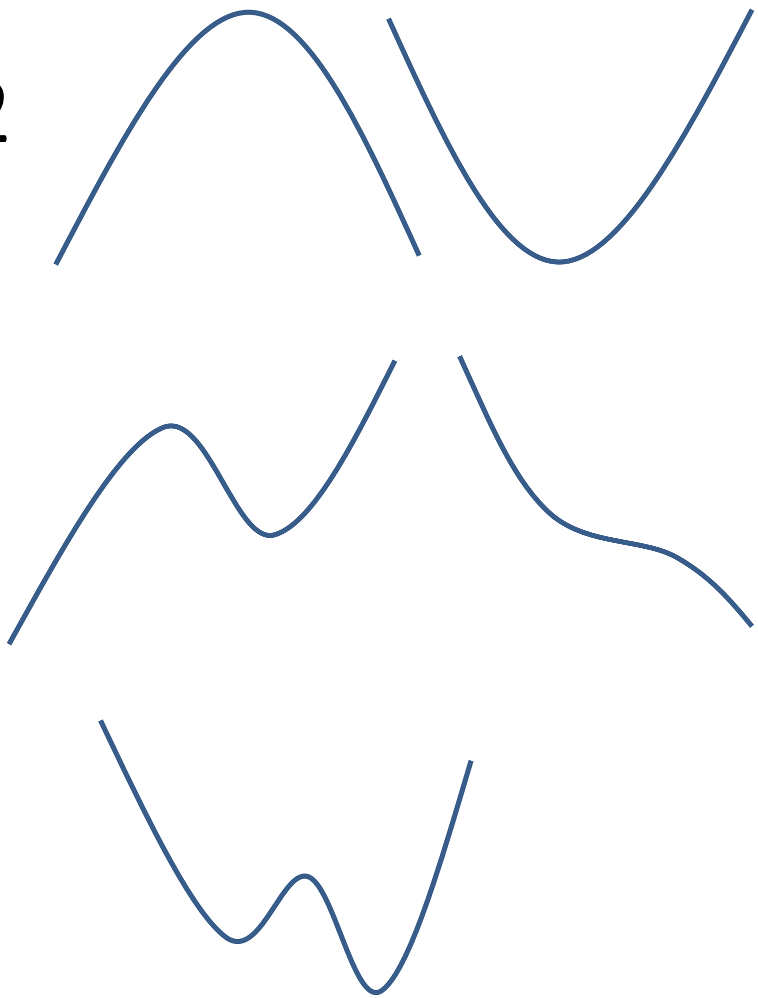
Polynomial Graphs

Order:

2

3

4



What property connects the order of the polynomial and the shape?

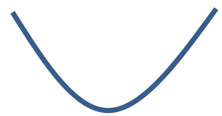
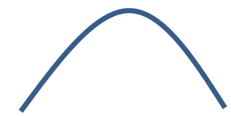
?

In Chapter 2 how did we tell what way up a quadratic is, and why does this work?

?

Polynomial Graphs

e.g. If $y = 2x^2 + 3$, try a large positive value like $x = 1000$. We can see we'd get a large positive y value. Thus as $x \to \infty$, $y \to \infty$

Equation	If $a > 0$	Resulting Shape	If $a < 0$	Resulting Shape
$y = ax^2 + bx + c$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to \infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to -\infty$	
$y = ax^3 + bx^2 + cx + d$	?		?	
$y = ax^4 + bx^3 + cx^2 + dx + e$	?		?	
$y = ax^5 + bx^4 + \cdots$	?		?	

If $a > 0$, what therefore can we say about the shape if:

- **The order is odd:** ?
- **The order is even:** ?

(And we have the opposite if $a < 0$)

Cubics

Sketch the curve with equation
$$y = (x-2)(1-x)(1+x)$$

Sketch the curve with equation
$$y = x^2(x-1)$$

Features you must consider:

Shape? ?

Fro Tip: No need to expand out the whole thing. Just mentally consider the x terms multiplied together.

Roots? ?

***y*-intercept?** ?

? Final sketch

Shape? ?

Roots? ?

This is sort of because the curve crosses at 0 then immediately crosses at 0 again!

***y*-intercept?** ?

? Final sketch

Cubics

Sketch the curve with equation
$$y = (2 - x)(x + 1)^2$$

Shape? ?

Roots? ?

y-intercept? ?

? Final sketch

Sketch the curve with equation
$$y = (x - 4)^3$$

Shape? ?

Roots? ?

y-intercept? ?

? Final sketch

Cubics with Limited Roots

Sketch the curve with equation

$$y = (x + 1)(x^2 + x + 1)$$

Shape? ?

Roots? ?

y-intercept? ?

? Final sketch

Finding the equation yourself :

Edexcel C1 May 2013(R) Q9

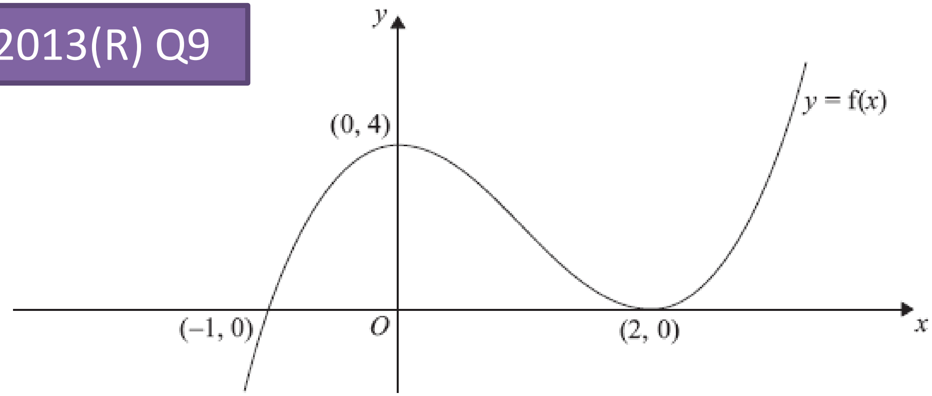


Figure 1 shows a sketch of the curve C with equation $y = f(x)$.
The curve C passes through the point (–1, 0) and touches the x-axis at the point (2, 0).
The curve C has a maximum at the point (0, 4).
The equation of the curve C can be written in the form.

$$y = x^3 + ax^2 + bx + c$$

where a, b and c are integers.
(a) Calculate the values of a, b, c.

?

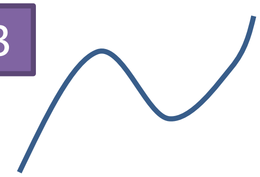
Test Your Understanding

1 Sketch the curve with equation
$$y = x(x-3)^2$$

?

2 Sketch the curve with equation
$$y = -(x+2)^3$$

?

3 

A curve has this shape, touches the x axis at 3 and crosses the x axis at -2. Give a suitable equation for this graph.

?

Sketch the curve with equation
$$y = 2x^2(x-1)(x+1)^3$$

?

(I took this question from my Riemann Zeta Club materials: www.drfrostmaths.com/rzc)

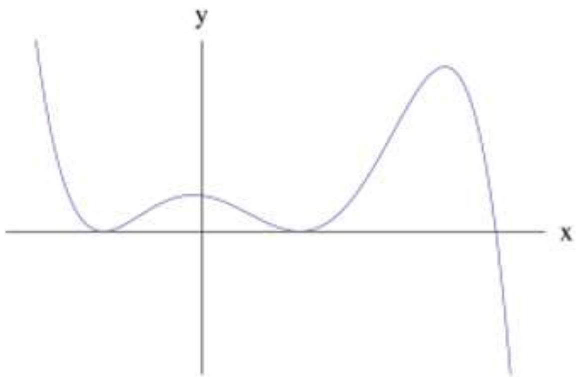
[end shameless plug]

Exercise 4A

Pearson Pure Mathematics Year 1/AS
Pages 62-63

Extension

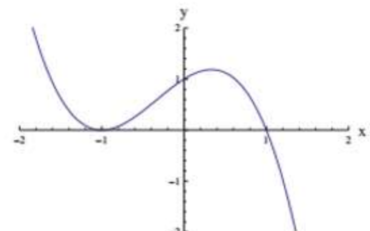
1. [MAT 2012 1E] Which one of the following equations could possibly have the graph given below?



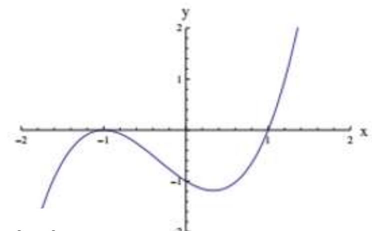
A) $y = (3 - x)^2(3 + x)^2(1 - x)$
B) $y = -x^2(x - 9)(x^2 - 3)$
C) $y = (x - 6)(x - 2)^2(x + 2)^2$
D) $y = (x^2 - 1)^2(3 - x)$

?

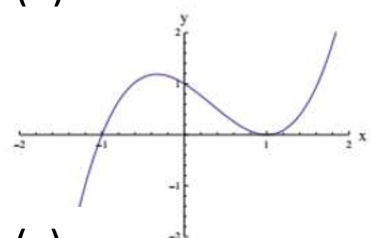
2. [MAT 2011 1A] A sketch of the graph $y = x^3 - x^2 - x + 1$ appears on which of the following axis?



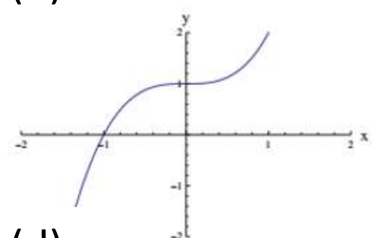
(a)



(b)



(c)



(d)

?

Recap

If we sketched $y = (x-a)(x-b)^2(x-c)^3$ what happens on the x-axis at:

$x = a$: ?

$x = b$: ?

$x = c$: ?

Quartics

If you understand the principle of sketching polynomials in general, then sketching quartics shouldn't feel like anything new.
Recall that if the x^4 term is positive, the 'tails' both go upwards, otherwise downwards.

Sketch the curve with equation
$$y = x(x+1)(x-2)(x-3)$$

Shape: ?
Roots: ?
y-intercept: ?

?

Sketch the curve with equation
$$y = (x-2)^2(x+1)(3-x)$$

Shape: ?
Roots: ?
y-intercept: ?

?

Quartics

Sketch the curve with equation
$y = (x+1)(x-1)^3$

Hint

?

Sketch the curve with equation
$y = (x-2)^4$

Hint

?

Test Your Understanding

Sketch the curve with equation
$$y = x^2(x+1)(x-1)$$

?

Sketch the curve with equation
$$y = -(x+1)(x-3)^3$$

?

Exercise 4B

Pearson Pure Mathematics Year 1/AS
Pages 65-66

Extension

1 *[STEP I 2012 Q2a]*

a. Sketch $y = x^4 - 6x^2 + 9$

b. For what values of b does the equation $y = x^4 - 6x^2 + b$ have the following number of distinct roots (i) 0, (ii) 1, (iii) 2, (iv) 3, (v) 4.

a) ?

b) By changing b, we shift the graph up and down. Then we can see that:

i) ?
ii) ?
iii) ?
iv) ?
v) ?

GCSE RECAP :: Reciprocal Graphs

Sketch $y = \frac{1}{x}$

?

Notice the distance between this line and the x-axis (i.e. the line $y = 0$) gradually decreases as the lines go off towards infinity. The line $y = 0$ is known as an **asymptote** of the graph.

Sketch $y = -\frac{3}{x}$

?

An asymptote is a line which the graph approaches but never reaches.

Asymptotes of $y = \frac{a}{x}$:

$y = 0$,

?

Reciprocal Graphs

Sketch $y = \frac{3}{x^2}$

This is new to the A Level 2017 syllabus.

Sketch $y = -\frac{4}{x^2}$

?

?

Hint: Note that anything squared will always be at least 0.

Reciprocal Graphs

On the same axes, sketch $y = \frac{1}{x}$ and $y = \frac{3}{x}$

?

Exercise 4C

Pearson Pure Mathematics Year 1/AS
Page 67

Points of Intersection

In the previous chapter we saw why the points of intersection of two graphs gave the solutions to the simultaneous equations corresponding to these graphs.

If $y = f(x)$ and $y = g(x)$, then the x values of the points of intersection can be found when $f(x) = g(x)$.

Example: On the same diagram sketch the curves with equations $y = x(x-3)$ and $y = x^2(1-x)$. Find the coordinates of their points of intersection.

Sketch

?

Further example involving unknown constants

On the same diagram sketch the curves with equations $y = x^2(3x - a)$ and $y = \frac{b}{x}$, where a, b are positive constants. State, giving a reason, the number of real solutions to the equation $x^2(3x - a) - \frac{b}{x} = 0$

?

?

?

Test Your Understanding

On the same diagram sketch the curves with equations $y = x(x-4)$ and $y = x(x-2)^2$, and hence find the coordinates of any points of intersection.

?

Hint: Remember you can use the discriminant to reason about the number of solutions of a quadratic.

Exercise 4D

Pearson Pure Mathematics
Year 1/AS, Pages 69-71

Extension

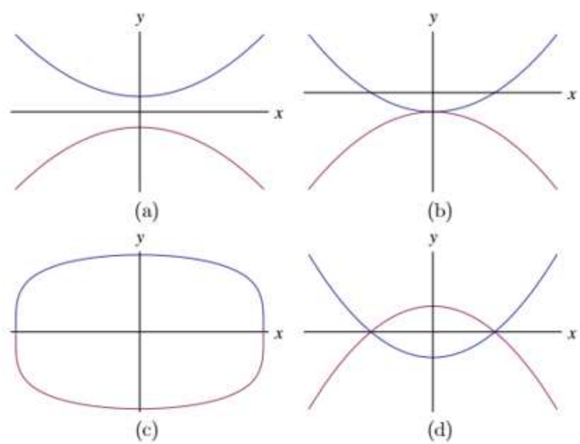
1 [MAT 2005 1B]
The equation $(x^2+1)^{10} = 2x - x^2 - 2$
A) has $x = 2$ as a solution;
B) has no real solutions;
C) has an odd number of real solutions;
D) has twenty real solutions.

?

2 [MAT 2010 1A] The values of k for which the line $y = kx$ intersects the parabola $y = (x-1)^2$ are precisely
A) $k \leq 0$
B) $k \geq -4$
C) $k \geq 0$ or $k \leq -4$
D) $-4 \leq k \leq 0$

?

3 [MAT 2013 1D]
Which of the following sketches is a graph of $x^4 - y^2 = 2y + 1$?



?

Transformations of Functions

Suppose $f(x) = x^2$

Sketch $y = f(x)$:

?

Then $f(x+2) = $?

Sketch $y = f(x+2)$

?

What do you notice about the relationship between the graphs of $y = f(x)$ and $y = f(x+2)$?

?

Transformations of Functions

This is all you need to remember when considering how transforming your function transforms your graph...

	Affects which axis?	What we expect or opposite?
Change **inside** $f(\,)$	?	?
Change **outside** $f(\,)$	?	?

Therefore...

$y = f(x-3)$ → ?

$y = f(x) + 4$ → ?

$y = f(5x)$ → ?

$y = 2f(x)$ → ?

Sketching transformed graphs

Sketch $y = x^2 + 3$

?

Sketch $y = \frac{2}{x+1}$

?

More Examples

Sketch $y = x(x+2)$. On the same axes, sketch $y = (x-a)(x-a+2)$, where $a > 2$.

?

Sketch $y = x^2(x-4)$. On the same axes, sketch the graph with equation

$$y = (2x)^2(2x-4)$$

?

Reflections of Graphs

If $y = x(x + 2)$, sketch $y = f(x)$ and $y = -f(x)$ on the same axes.

?

Test Your Understanding

If $y = (x+1)(x-2)$, sketch $y = f(x)$ and $y = f(\frac{x}{3})$ on the same axes.

?

Sketch the graph of $y = \frac{2}{x} + 1$, ensuring you indicate any intercepts with the axes.

?

Exercise 4E/4F

Pearson Pure Mathematics Year 1/AS
Pages 74-75 (translations), 78 (stretches/reflections)

Effect of transformation on specific points

Sometimes you will not be given the original function, but will be given a sketch with specific points and features you need to transform.
Where would each of these points end up?

$y = f(x)$	$(4, 3)$	$(1, 0)$	$(6, -4)$
$y = f(x+1)$	?	?	?
$y = f(2x)$	?	?	?
$y = 3f(x)$	?	?	?
$y = f(x) - 1$	?	?	?
$y = f\left(\frac{x}{4}\right)$	?	?	?
$y = f(-x)$	?	?	?
$y = -f(x)$	?	?	?

Test Your Understanding

Edexcel C1 May 2012 Q10

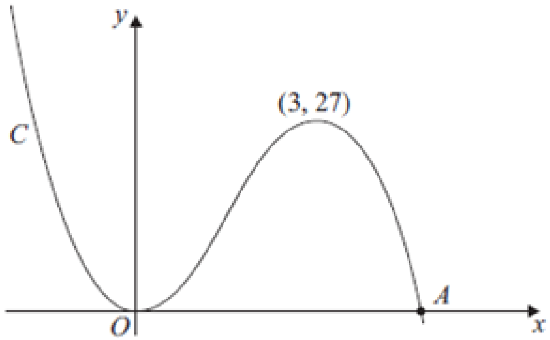


Figure 1

Figure 1 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^2(9 - 2x).$$

There is a minimum at the origin, a maximum at the point (3, 27) and C cuts the x-axis at the point A.

(*a*) Write down the coordinates of the point A.

(1)

(*b*) On separate diagrams sketch the curve with equation

(i) $y = \mathrm{f}(x + 3)$,

(ii) $y = \mathrm{f}(3x)$.

On each sketch you should indicate clearly the coordinates of the maximum point and any points where the curves cross or meet the coordinate axes.

(6)

The curve with equation $y = \mathrm{f}(x) + k$, where k is a constant, has a maximum point at (3, 10).

(*c*) Write down the value of k.

(1)

(a) ?

(b)(i) ?

(ii) ?

(c) ?

Exercise 4G

Pearson Pure Mathematics Year 1/AS

Pages 80-81