Upper 6 Chapter 10

Numerical Methods

Chapter Overview

1. Locating Roots

2. Iteration

3. The Newton-Raphson Method

4. Applications to Modelling

9.1	Locate roots of $f(x) = 0$ by considering changes of sign of $f(x)$ in an interval of x on which $f(x)$ is sufficiently well behaved. Understand how change of sign methods can fail.	Students should know that sign change is appropriate for continuous functions in a small interval. When the interval is too large sign may not change as there may be an even number of roots. If the function is not continuous, sign may change but there may be an asymptote (not a root).
9.2	Solve equations approximately using simple iterative methods; be able to draw associated cobweb and staircase diagrams.	Understand that many mathematical problems cannot be solved analytically, but numerical methods permit solution to a required level of accuracy. Use an iteration of the form $x_{n+1} = f(x_n)$ to find a root of the equation $x = f(x)$ and show understanding of the convergence in geometrical terms by drawing cobweb and staircase diagrams.
9.3	Solve equations using the Newton-Raphson method and other recurrence relations of the form $x_{n+1} = g(x_n)$ Understand how such methods can fail.	For the Newton-Raphson method, students should understand its working in geometrical terms, so that they understand its failure near to points where the gradient is small.

9.4	Understand and use numerical integration of functions, including the use of the trapezium rule and estimating the approximate area under a curve and limits that it must lie between.	For example, evaluate $\int_0^1 \sqrt{(2x+1)}\ \mathrm{d}x$ using the values of $\sqrt{(2x+1)}$ at $x = 0$, 0.25, 0.5, 0.75 and 1 and use a sketch on a given graph to determine whether the trapezium rule gives an over-estimate or an under-estimate.
9.5	Use numerical methods to solve problems in context.	Iterations may be suggested for the solution of equations not soluble by analytic means.

LOCATING ROOTS

Finding the root of a function $f(x)$ is to **solve the equation** $\boldsymbol{f(x) = 0}$

However, for some functions, the 'exact' root is complicated and difficult to calculate …

For example:

$$\boldsymbol{x^3 + 2x^2 - 3x + 4 = 0}$$

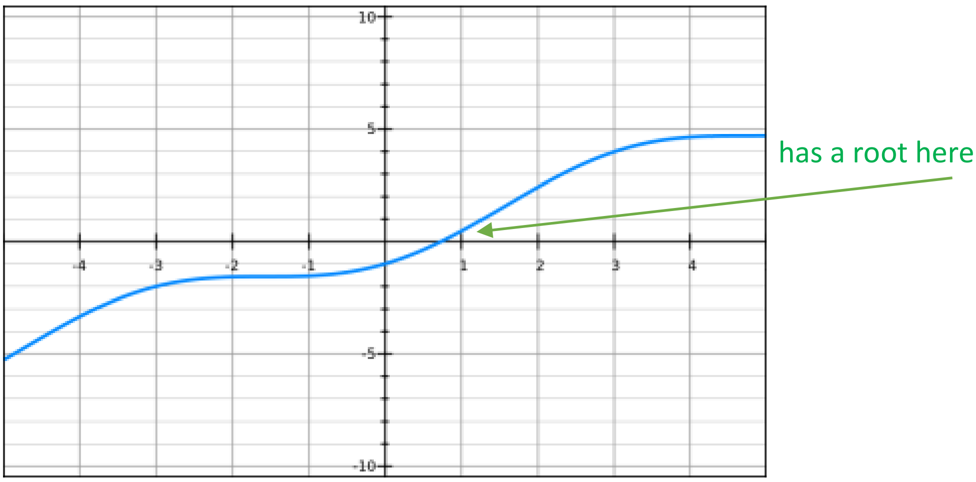
has the solution:

$$x = \frac{1}{3}\left(-2 - \frac{13}{\sqrt[3]{89 - 6\sqrt{159}}} - \sqrt[3]{89 - 6\sqrt{159}}\right)$$

… or there is no 'algebraic' expression at all. (involving roots, logs, sin, cos, etc.)

For example:

$$\boldsymbol{x - \cos(x) = 0}$$



To show that a root exists in a given interval, show that $\boldsymbol{f(x)}$ changes sign

Example 1

Show that $f(x) = e^x + 2x - 3$ has a root between $x = 0.5$ and $x = 0.6$

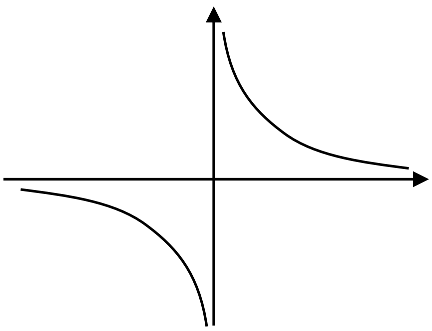
STEP 1: Find $f(x)$ for the two values given in the question

STEP 2: Write a concluding statement referring to the change in sign and the fact that $f(x)$ is a continuous function

Note on functions that are NOT continuous:

If the function is not continuous, the sign change may be due to an asymptote rather than a root.

For example:



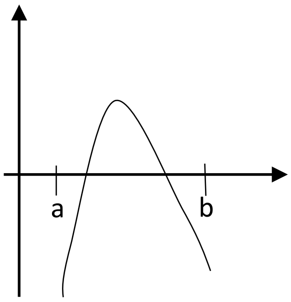
When $f(x) = \frac{1}{x}$, then $f(-1) = -1$ and $f(1) = 1$.

However, although there is a sign change, a root does not exist between $x = -1$ and $x = 1$

Note on continuous functions:

A continuous function could simply have an even number of roots in a given interval rather than no roots.

For example:



Here $f(a)$ is negative and $f(b)$ is also negative

However, although there are two roots, a sign change does not occur.

Example 2 Edexcel C3 Jan 2013

$$g(x) = e^{x-1} + x - 6$$

The root of $g(x) = 0$ is α.

(c) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places.

(3)

Example 3

(a) Using the same axes, sketch the graphs of $y = \ln x$ and $y = \frac{1}{x}$. Explain how your diagrams shows that the function $y = \ln(x) - \frac{1}{x}$ has only one root.

(b) Show that this root lies in the interval $1.7 < x < 1.8$

(c) Given that the root of $f(x)$ is α, show that $\alpha = 1.763$ correct to 3 decimal places.

Exercise 10A Page 276

ITERATION

To solve $f(x) = 0$ by an iterative method, rearrange into a form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$

Example 1 Edexcel C3 Jan 2013

$$g(x) = e^{x-1} + x - 6$$

(*a*) Show that the equation $g(x) = 0$ can be written as

$$x = \ln(6 - x) + 1, \qquad x < 6.$$

(2)

The root of $g(x) = 0$ is α.

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \qquad x_0 = 2.$$

is used to find an approximate value for α.

(*b*) Calculate the values of x_1, x_2 and x_3 to 4 decimal places.

(3)

(*c*) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places.

(3)

a)

b) **x_1, x_2, x_3 represent successively better approximations of the root**

Initially type x_0 (i.e. 2) onto your calculator.
Now just type: $\ln(6 - ANS) + 1$
And then press your $=$ key to get successive iterations.

c)

The starting value $x_0$matters.

- If there are a multiple roots, the iteration might converge to (i.e. approach) a different root.
- The iteration not converge to a root at all and **diverges** (i.e. approach infinity).

Example 2

$$f(x) = x^3 - 3x^2 - 2x + 5$$

(a) Show that the equation $f(x) = 0$ has a root in the interval $3 < x < 4$.

(b) Use the iterative formula $x_{n+1} = \sqrt{\frac{x_n^3 - 2x_n + 5}{3}}$ to calculate the values of x_1, x_2 and x_3, giving your answers to 4 decimal places, and taking:
(i) $x_0 = 1.5$ (ii) $x_0 = 4$

Staircase and cobweb diagrams

Example 3

$$f(x) = x^2 - 8x + 4$$

(a) Show that the root of the equation $f(x) = 0$ can be written as $x = \sqrt{8x - 4}$

(b) Using the iterative formula $x_{n+1} = \sqrt{8x_n - 4}$, and starting with $x_0 = 1$, draw a staircase diagram, indicating $x_0, x_1,\ x_2$ on your x-axis, as well as the root α.

Exercise 10B Page 280

THE NEWTON-RAPHSON METHOD

The Newton- Raphson method can be used to find numerical solutions to equations of the form $f(x) = 0$. You need to be able to differentiate $f(x)$ in order to use this method.

The Newton- Raphson formula is:

$$\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Example 1

Recall that in lesson 1 we saw that the function $\boldsymbol{f(x) = x - \cos(x)}$ has a root, α, in the interval $0 < \alpha < 1$.

Using $x_0 = 0.5$ as a first approximation to α, apply the Newton-Raphson procedure three times to find a better approximation to α which, in this case, will be accurate to 7 decimal places.

To perform iterations quickly, do the following on your calculator:

[0.5] [=]

[ANS] – (ANS – cos(ANS))/(1 + sin(ANS))

Then press [=].

Example 2 Edexcel FP1 June 2013(R) Q3c

$$f(x) = \frac{1}{2}x^4 - x^3 + x - 3$$

...

The equation $f(x) = 0$ has a root β in the interval $[-2, -1]$.

(*c*) Taking -1.5 as a first approximation to β, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to β.
Give your answer to 2 decimal places.

(5)

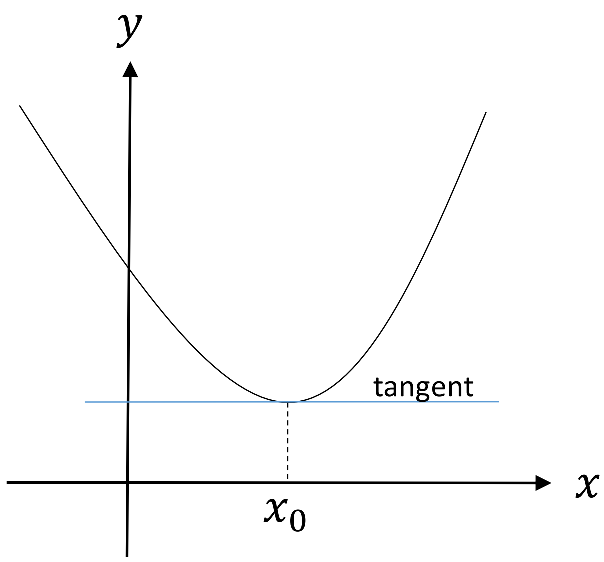
Example 3 Edexcel FP1 Jan 2010 Q2c

$$f(x) = 3x^2 - \frac{11}{x^2}.$$

(*c*) Taking 1.4 as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to obtain a second approximation to α, giving your answer to 3 decimal places.

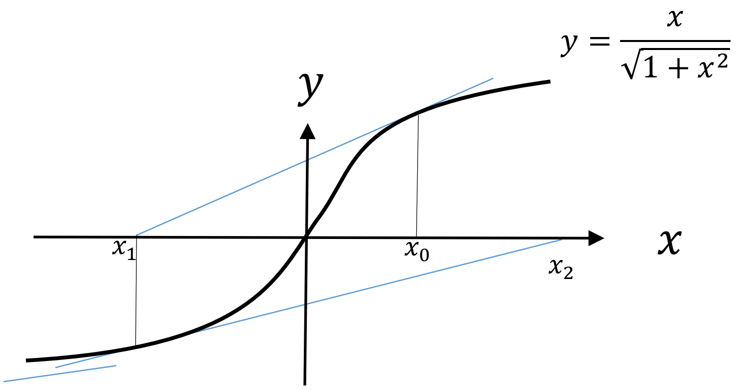
(5)

When does Newton-Raphson fail?



$$\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

If the starting value x_0 was the stationary point, then $f'(x_0) = 0$, resulting in a division by 0 in the above formula.
Graphically, it is because the tangent will never reach the x-axis.



Newton-Raphson also suffers from the same drawbacks as solving by iteration, in that it is possible for the values of x_i to **diverge**.
In this example, the x_i oscillate either side of 0, but get gradually further away from $\alpha = 0$.

Exercise 10C Page 284

APPLICATIONS TO MODELLING

Example 4

The price of a car in £s, x years after purchase, is modelled by the function

$$f(x) = 15\,000\,(0.85)^x - 1000\sin x\,, \quad x > 0$$

(a) Use the model to find the value, to the nearest hundred £s, of the car 10 years after purchase.

(b) Show that $f(x)$ has a root between 19 and 20.

(c) Find $f'(x)$

(d) Taking 19.5 as a first approximation, apply the Newton-Raphson method once to $f(x)$ to obtain a second approximation for the time when the value of the car is zero. Give your answer to 3 decimal places.

(e) Criticise this model with respect to the value of the car as it gets older.

Exercise 10D Page 287