P2 Chapter 4 :: Binomial Expansion

jfrost@tiffin.kingston.sch.uk

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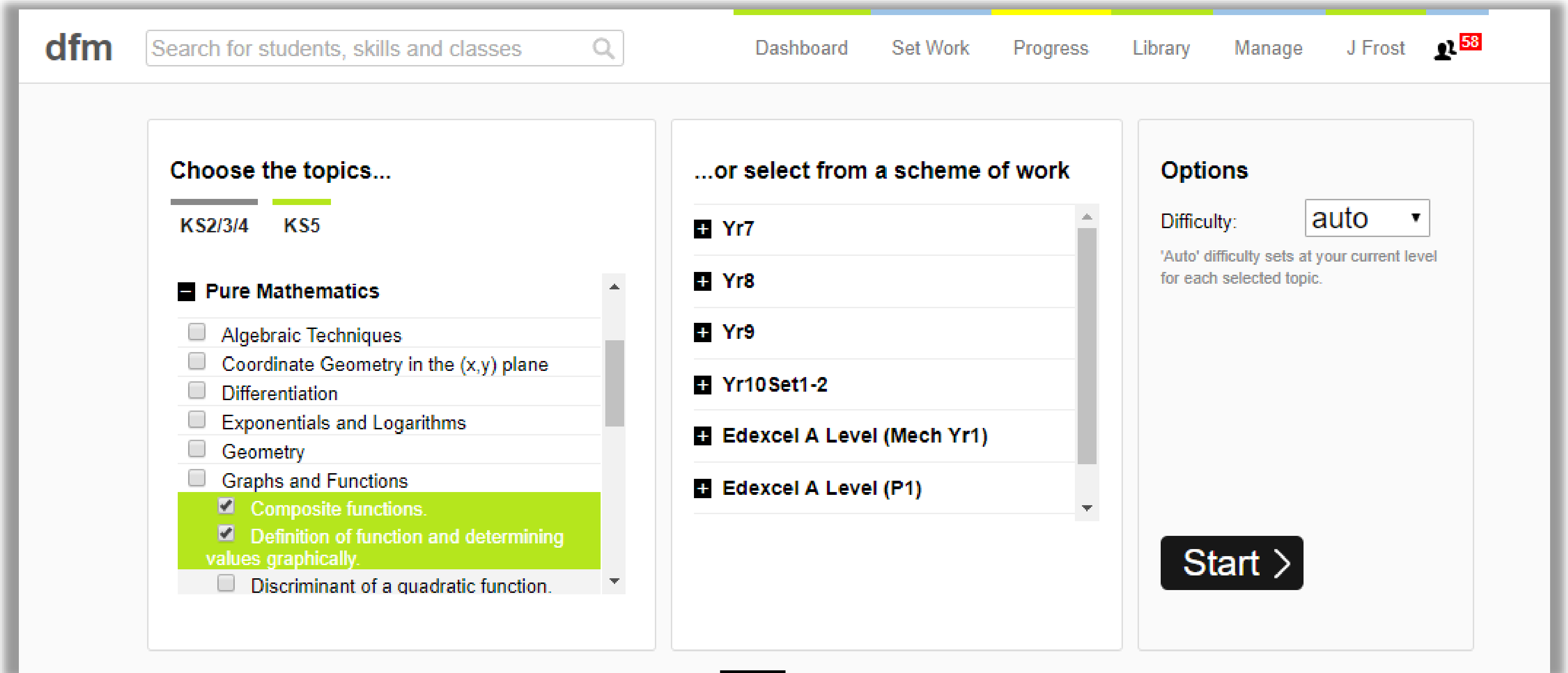

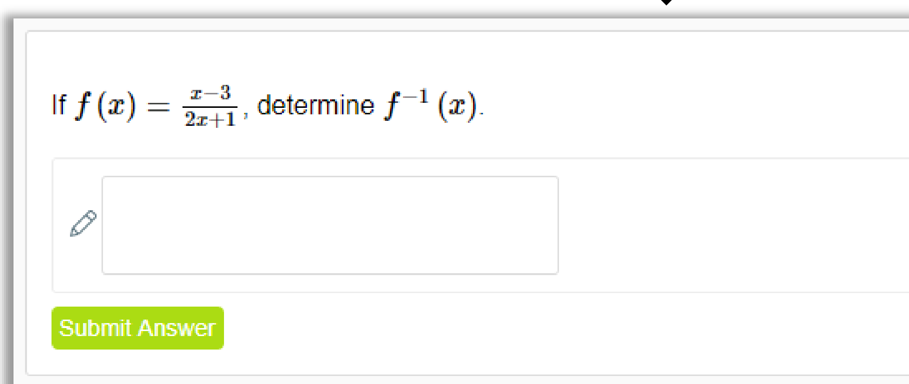

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Practise questions by chapter, including past paper Edexcel questions and extension questions (e.g. MAT).

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Chapter Overview

In Year 1 you found the Binomial expansion of $(a+b)^n$ where n was a positive integer. This chapter allows you to extend this to when n is any rational number, i.e. could be negative or fractional.

1:: Binomial Expansion for negative/fractional powers.

"Expand $\sqrt{1+x}$ in ascending powers of x up to the x^2 term."

2:: Constant is not 1.

The same, but where the term preceding the x is not 1, e.g.

"Expand $(8+5x)^{-\frac{1}{3}}$ in ascending powers of x up to the x^3 term."

3:: Using Partial Fractions

"Show that the cubic approximation of $\frac{4-5x}{(1+x)(2-x)}$ is $2-\frac{7}{2}x+\frac{11}{4}x^2-\frac{25}{8}x^3$"

Pure Year 1 Recap

Remember that for small integer n you could use a row of Pascal Triangle for the Binomial coefficients, descending powers of the first term and ascending powers of the second.
If the first term is 1, we can ignore the powers of 1.

$(1+x)^5 =$?

$(1+2x)^4 =$?

$=$

$(1-3x)^3 =$?

$=$

Do you remember the simple way to find your Binomial coefficients?

Hopefully you can see the pattern by this point.

$\binom{n}{1} =$? $\binom{n}{2} =$? $\binom{n}{3} =$? $\binom{n}{4} =$?

$\binom{10}{3} =$? $\binom{-1}{2} =$? $\binom{-2}{3} =$?

$\binom{0.5}{2} =$?

Note: You can work out a 'choose' value, in the same way, when the top number is negative or fractional, but your calculator can not do this directly.

Pure Year 1 Recap

$$(a+b)^n = a^n + nC1\ a^{n-1}b + nC2\ a^{n-2}b^2 + \cdots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + {}^nC_r\, x^n$$

Expand $(1+x)^{11}$ up to and including the term in x^3

?

Expand $(1-2x)^8$ up to and including the term in x^3

?

Binomial Expansion for any Rational Index

It can be shown that the following result is also true for any rational value of n provided certain conditions are met.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + {}^nC_r\, x^n$$

Use the binomial expansion to find the first four terms of $\frac{1}{1+x}$

?

And the first four terms of $\sqrt{1-3x}$

?

When are infinite expansions valid?

Our expansion might be an infinite number of terms. If so, the result must converge

$$\frac{1}{1+x} = (1+x)^{-1}$$
$$= 1 + (-1)x + \frac{-1 \times -2}{2!}x^2 + \frac{-1 \times -2 \times -3}{3!}x^3 + \cdots$$
$$= \mathbf{1 - x + x^2 - x^3 + \cdots}$$

What would happen in the expansion if:

a) $x > 1$: ?

b) $0 < x < 1$: ?

c) $-1 < x < 0$: ?

d) $x = 1$: ?

Therefore requirement on x:

?

1. The modulus of a number

The modulus of a number is its absolute size. That is, we disregard any sign it might have.

Example

The modulus of -8 is simply 8.

The modulus of $-\frac{1}{2}$ is $\frac{1}{2}$.

The modulus of 17 is simply 17.

The modulus of 0 is 0.

So, the modulus of a positive number is simply the number.

The modulus of a negative number is found by ignoring the minus sign.

The modulus of a number is denoted by writing vertical lines around the number.

Note also that the modulus of a negative number can be found by multiplying it by -1 since, for example, $-(-8) = 8$.

This observation allows us to define the modulus of a number quite concisely in the following way

$$|x| = \begin{cases} x & \text{if } x \text{ is positive or zero} \\ -x & \text{if } x \text{ is negative} \end{cases}$$

Example

$$|9| = 9, \qquad |-11| = 11, \qquad |0.25| = 0.25, \qquad |-3.7| = 3.7$$

When are infinite expansions valid?

Expansions are allowed to infinite. However, the result must converge

$$\sqrt{1-3x} = (1-3x)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}(-3x) + \frac{\frac{1}{2} \times -\frac{1}{2}}{2!}(-3x)^2 + \frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{3!}(-3x)^3 + \cdots$$
$$= \mathbf{1 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 - \cdots}$$

This time, what do you think needs to be between -1 and 1 for the expansion to be valid?

?

✎ An infinite expansion $(1+x)^n$ is valid if $|x| < 1$

Quickfire Examples:

Expansion of $(1+2x)^{-1}$ valid if: ?

Expansion of $(1-x)^{-2}$ valid if: ?

Expansion of $\left(1+\frac{1}{4}x\right)^{\frac{1}{2}}$ valid if: ?

Expansion of $\left(1-\frac{2}{3}x\right)^{-1}$ valid if: ?

Combining Expansions

Edexcel C4 June 2013 Q2

(*a*) Use the binomial expansion to show that

$$\sqrt{\left(\frac{1+x}{1-x}\right)} \approx 1 + x + \frac{1}{2}x^2, \qquad |x| < 1$$

(6)

Firstly express as a product:

?

How many terms do we need in each expansion?

?

Completing:

?

Test Your Understanding

Find the binomial expansion of $\frac{1}{(1+4x)^2}$ up to an including the term in x^3.

State the values of x for which the expansion is valid.

?

C4 Edexcel Jan 2010

?

1. (*a*) Find the binomial expansion of

$$\sqrt{(1-8x)}, \qquad |x| < \frac{1}{8},$$

in ascending powers of x up to and including the term in x^3, simplifying each term.

(6)

(*b*) Show that, when $x = \frac{1}{100}$, the exact value of $\sqrt{(1-8x)}$ is $\frac{\sqrt{23}}{5}$.

(2)

(*c*) Substitute $x = \frac{1}{100}$ into the binomial expansion in part (*a*) and hence obtain an approximation to $\sqrt{23}$. Give your answer to 5 decimal places.

(3)

Accuracy of an approximation

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

If $x = 0.01$, how accurate would the approximation $1 - x + x^2$ by for the value of $\frac{1}{1+x}$?

?

Exercise 4A

Pearson Pure Mathematics Year 2/AS
Pages 96-97

Extension

1 [STEP I 2011 Q6] Use the binomial expansion to show that the coefficient of x^r in the expansion of $(1-x)^{-3}$ is $\frac{1}{2}(r+1)(r+2)$.

(i) Show that the coefficient of x^r in the expansion of $\frac{1-x+2x^2}{(1-x)^3}$ is r^2+1 and hence find the sum of the series

$$1+\frac{2}{2}+\frac{5}{4}+\frac{10}{8}+\frac{17}{16}+\frac{26}{32}+\frac{37}{64}+\cdots$$

(ii) Find the sum of the series

$$1+2+\frac{9}{4}+2+\frac{25}{16}+\frac{9}{8}+\frac{49}{64}$$

Common Errors

$$\sqrt{1-3x} = (1-3x)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}(-3x) + \frac{\frac{1}{2} \times -\frac{1}{2}}{2!}(-3x)^2 + \frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{3!}(-3x)^3 + \cdots$$
$$\boldsymbol{= 1 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 - \cdots}$$

What errors do you think are easy to make?

- Sign errors, e.g. $(-3x)^2 = -9x^2$
- Not putting brackets around the $-3x$, e.g. $-3x^2$ instead of $(-3x)^2$
- Dividing by say 3 instead of 3!

Dealing with $(a + bx)^n$

Find first four terms in the binomial expansion of $\sqrt{4+x}$
State the values of x for which the expansion is valid.

$(4+x)^{\frac{1}{2}} =$?

$=$?

$=$?

$=$?

$=$?

Valid if ?

We need it in the form $(1+x)^n$. So factorise the 4 out.

Remember for the expansion of $(1+x)^n$ to be valid then $|x| < 1$. In this case the 'x' is in fact $\frac{1}{4}x$.

Quickfire First Step

What would be the first step in finding the Binomial expansion of each of these?

Binomial expansion valid if:

$(2+x)^{-3} \Rightarrow$	?	?
$(9+2x)^{\frac{1}{2}} \Rightarrow$	?	?
$(8-x)^{\frac{1}{3}} \Rightarrow$	?	?
$(5-2x)^{-3} \Rightarrow$	?	?
$(16+3x)^{-\frac{1}{2}}$	?	?

Test Your Understanding

Edexcel C4 June 2013 (Withdrawn) Q1

(a) Find the binomial expansion of

$$\sqrt{(9+8x)}, \quad |x| < \frac{9}{8}$$

in ascending powers of x, up to and including the term in x^2.
Give each coefficient as a simplified fraction.

(5)

(b) Use your expansion to estimate the value of $\sqrt{(11)}$, giving your answer as a single fraction.

(3)

(a) ?

B1

M1;

A1 $\sqrt{}$

(b) ?

A1 oe

A1

Exercise 4B

Pearson Pure Mathematics Year 2/AS
Pages 99-100

Extension

[AEA 2006 Q1]

(a) For $|y| < 1$, write down the binomial series expansion of $(1 - y)^{-2}$ in ascending powers of y up to and including the term in y^3.

(b) Hence, or otherwise, show that
$$1 + \frac{2x}{1+x} + \frac{3x^2}{(1+x)^2} + \cdots + \frac{rx^{r-1}}{(1+x)^{r-1}} + \cdots$$
can be written in the form $(a + x)^n$. Write down the values of the integers a and n.

(c) Find the set of values of x for which the series in part (b) is convergent.

(a) ? a

(b) ? b

(c) ? c

Using Partial Fractions

Partial fractions allows us to split up a fraction into ones we can then find the binomial expansion of.

[Textbook]

a) Express $\frac{4-5x}{(1+x)(2-x)}$ as partial fractions.

b) Hence show that the cubic approximation of $\frac{4-5x}{(1+x)(2-x)}$ is $2-\frac{7}{2}x+\frac{11}{4}x^2-\frac{25}{8}x^3$

c) State the range of values of x for which the expansion is valid.

a

?

b

?

c

?

Test Your Understanding

[C4 June 2010 Q5]

10.
$$\frac{2x^2+5x-10}{(x-1)(x+2)} \equiv A + \frac{B}{x-1} + \frac{C}{x+2}.$$

(*a*) Find the values of the constants A, B and C. **(4)**

(*b*) Hence, or otherwise, expand $\dfrac{2x^2+5x-10}{(x-1)(x+2)}$ in ascending powers of x, as far as the term in x^2. Give each coefficient as a simplified fraction. **(7)**

? a

? b

Exercise 4C

Pearson Pure Mathematics Year 2/AS
Pages 102-103