P1 Chapter 10 :: Trigonometric Identities & Equations

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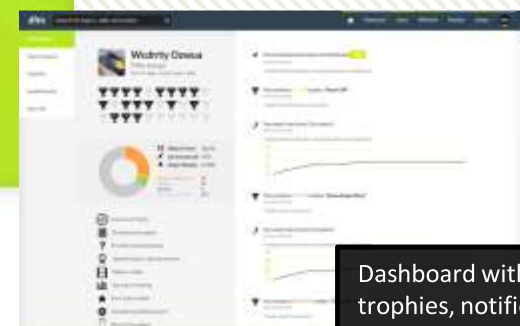

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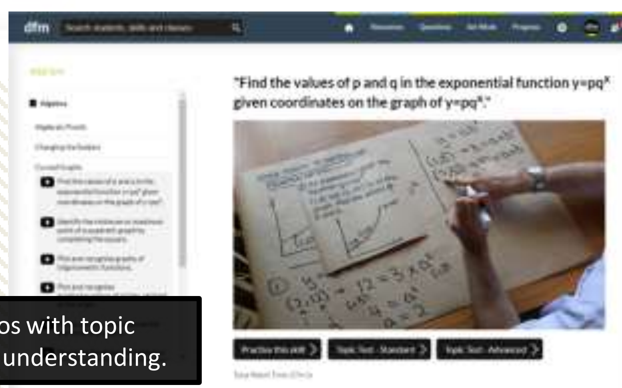

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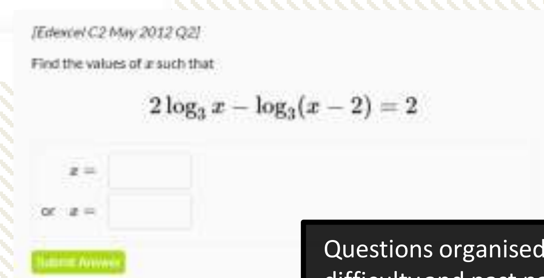


Questions organised by topic, difficulty and past paper.

Chapter Overview

Those who did IGCSE Further Maths or Additional Maths will be familiar with this content. Exact trigonometric values for $30°, 45°, 90°$ were in the GCSE syllabus.

1:: Know exact trig values for $30°, 45°, 60°$ and understand unit circle.

2:: Use identities $\frac{\sin x}{\cos x} \equiv \tan x$ and $\sin^2 x + \cos^2 x \equiv 1$

Show that $3\sin^2 x + 7\sin x = \cos^2 x - 4$ can be written in the form
$4\sin^2 x + 7\sin x + 3 = 0$

3:: Solve equations of the form $\sin(n\theta) = k$ and $\sin(\theta \pm \alpha) = k$

Solve $\sin(2x)(\cos 2x + 1) = 0$, for $0 \leq x < 360°$.

4:: Solve equations which are quadratic in sin/cos/tan.

Solve, for $0 \leq x < 360°$, the equation

$$4\sin^2 x + 9\cos x - 6 = 0$$

sin/cos/tan of 30°, 45°, 60°

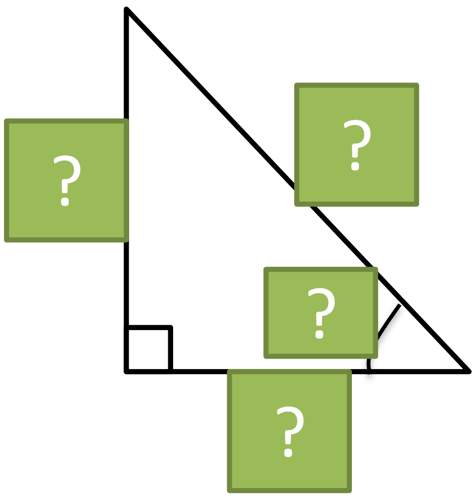
You will frequently encounter angles of 30°, 60°, 45° in geometric problems. Why?

?

Although you will always have a calculator, you need to know how to derive these.

All you need to remember:

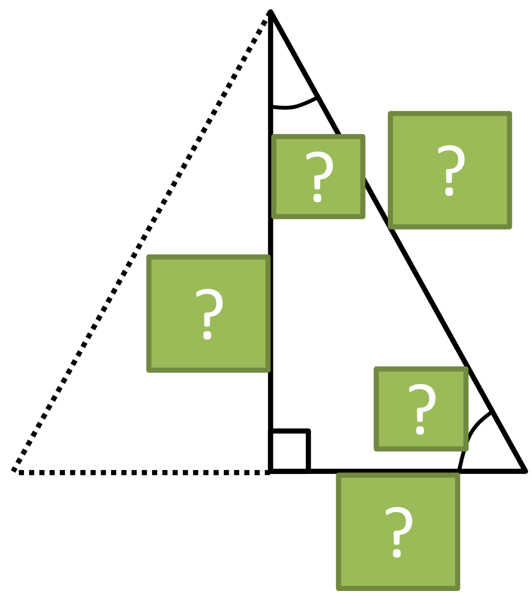
✎ **Draw half a unit square and half an equilateral triangle of side 2.**



$\sin(45°) = ?$

$\cos(45°) = ?$

$\tan(45°) = ?$



$\sin(30°) = ?$

$\cos(30°) = ?$

$\tan(30°) = ?$

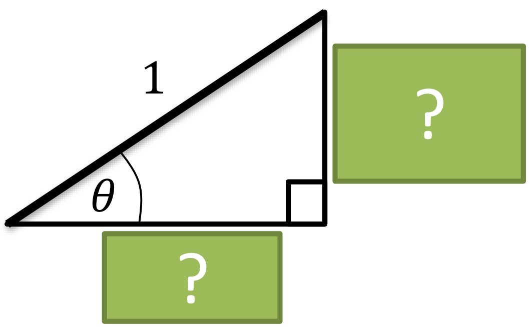
$\sin(60°) = ?$

$\cos(60°) = ?$

$\tan(60°) = ?$

The Unit Circle and Trigonometry

For values of θ in the range $0 < \theta < 90°$, you know that $\sin\theta$ and $\cos\theta$ are lengths on a right-angled triangle:

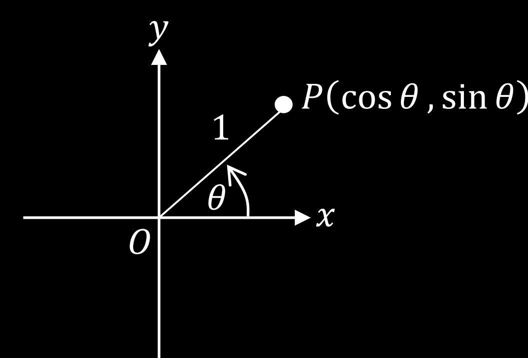


And what would be the **gradient** of the bold line?

?

But how do we get the rest of the graph for sin, cos and tan when $90° \leq \theta \leq 360°$?

✐ The point P on a unit circle, such that OP makes an angle θ with the positive x-axis, has coordinates $(\cos\theta, \sin\theta)$.
OP has gradient $\tan\theta$.



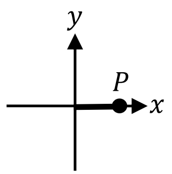
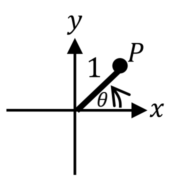
Angles are always measured **anticlockwise**.
(Further Mathematicians will encounter the same when they get to Complex Numbers)

We can consider the coordinate $(\cos\theta, \sin\theta)$ as θ increases from 0 to $360°$…

Mini-Exercise

Use the unit circle to determine each value in the table, **using either “0”, “+ve”, “-ve”, “1”, “-1” or “undefined”**.
Recall that the point on the unit circle has coordinate $(\cos\theta, \sin\theta)$ and OP has gradient $\tan\theta$.

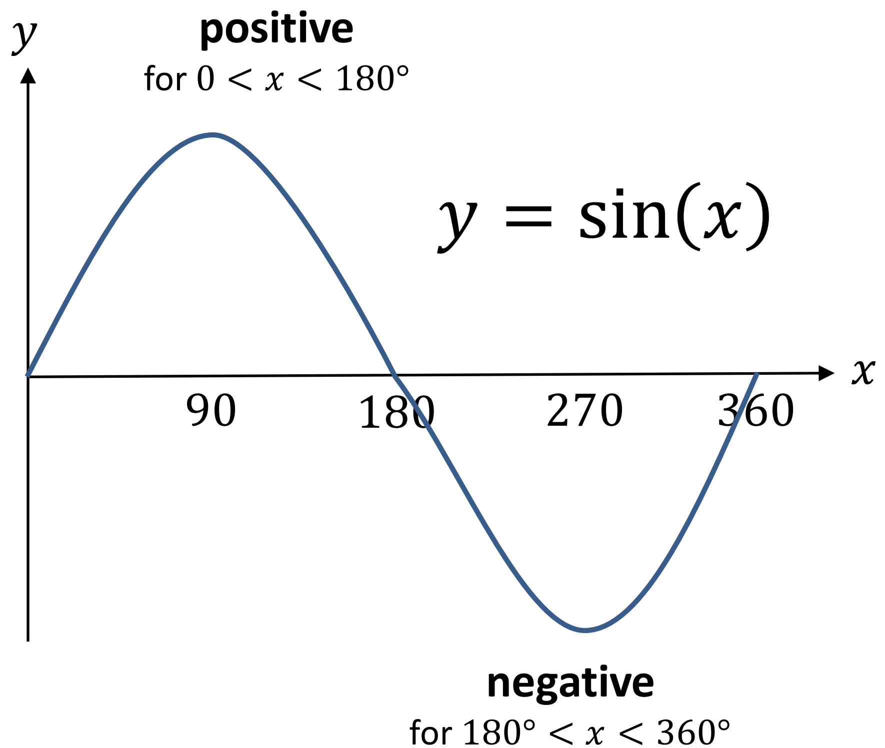
x-value | y-value | Gradient of OP.

	$\cos\theta$	$\sin\theta$	$\tan\theta$
$\theta = 0$ 	1	0	0
$0 < \theta < 90°$ 	?		
$\theta = 90°$?	?		
$90° < \theta < 180°$?	?		

	$\cos\theta$	$\sin\theta$	$\tan\theta$
$\theta = 180°$?	?		
$180° < \theta < 270°$?	?		
$\theta = 270°$?	?		
$270° < \theta < 360°$?	?		

The Unit Circle and Trigonometry

The unit circles explains the behaviour of these trigonometric graphs beyond $90°$. However, the easiest way to remember whether $\sin(x)$, $\cos(x)$, $\tan(x)$ are positive or negative is to just do a **very quick sketch (preferably mentally!)** of the corresponding graph.



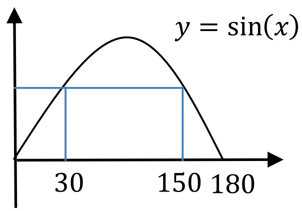
Note: The textbook uses something called '*CAST diagrams*'. I will not be using them in these slides, but you may wish to look at these technique as an alternative approach to various problems in the chapter.

A Few Trigonometric Angle Laws

The following are all easily derivable using a quick sketch of a trigonometric graph, and are merely a <u>convenience</u> so you don't always have to draw out a graph every time.
You are highly encouraged to **memorise these** so that you can do exam questions faster.

1 $\sin(x) = \sin(180° - x)$

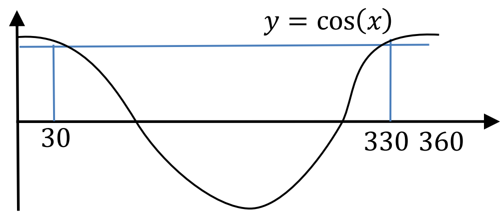
e.g. $\sin(150°) = \sin(30°)$



We saw this in the previous chapter when covering the 'ambiguous case' when using the sine rule.

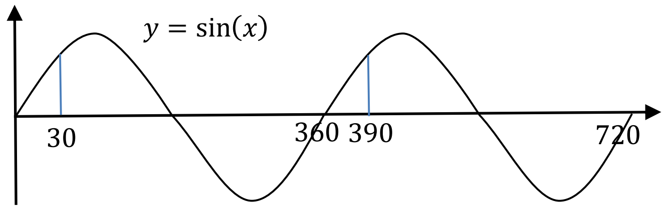
2 $\cos(x) = \cos(360° - x)$

e.g. $\cos(330°) = \cos(30°)$



3 sin and cos repeat every 360° but tan every 180°

e.g. $\sin(390°) = \sin(30°)$



4 $\sin(x) = \cos(90° - x)$

e.g. $\sin(50°) = \cos(40°)$

Remember from the previous chapter that "cosine" by definition is the sine of the "complementary" angle. This was/is never covered in the textbook but caught everyone by surprise when it came up in a C3 exam.

Examples

Without a calculator, work out the value of each below.

$\tan(225°) =$?

$\tan(210°) =$?

$\sin(150°) =$?

$\cos(300°) =$?

$\sin(-45°) =$?

$\cos(750°) =$?

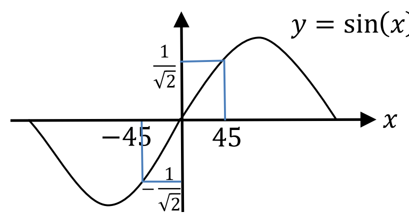
$\cos(120°) =$?

tan repeats every $180°$ so can subtract $180°$

For sin we can subtract from $180°$.

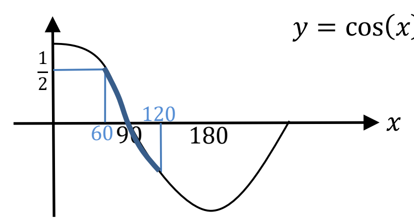
For cos we can subtract from $360°$.

We have to resort to a sketch for this one.



cos repeats every $360°$.

Again, let's just use a graph.



Use the 'laws' where you can, but otherwise just draw out a quick sketch of the graph.

- $\sin(x) = \sin(180 - x)$
- $\cos(x) = \cos(360 - x)$
- sin, cos repeat every $360°$ but tan every $180°$

Froflections: It's not hard to see from the graph that in general, $\sin(-x) = -\sin(x)$. Even more generally, a function f is known as an '**odd function**' if $f(-x) = -f(x)$. tan is similarly 'odd' as $\tan(-x) = -\tan(x)$.

A function is **even** if $f(-x) = f(x)$. Examples are $f(x) = \cos(x)$ and $f(x) = x^2$ as $\cos(-x) = \cos(x)$ and $(-x)^2 = (x)^2$. You do not need to know this for the exam.

The graph is rotationally symmetric about $90°$. Since $120°$ is $30°$ above $90°$, we get the same y value for $90° - 30° = 60°$, except negative.

Test Your Understanding

Without a calculator, work out the value of each below.

$\cos(315°) =$?

$\sin(420°) =$?

$\tan(-120°) =$?

$\tan(-45°) =$?

$\sin(135°) =$?

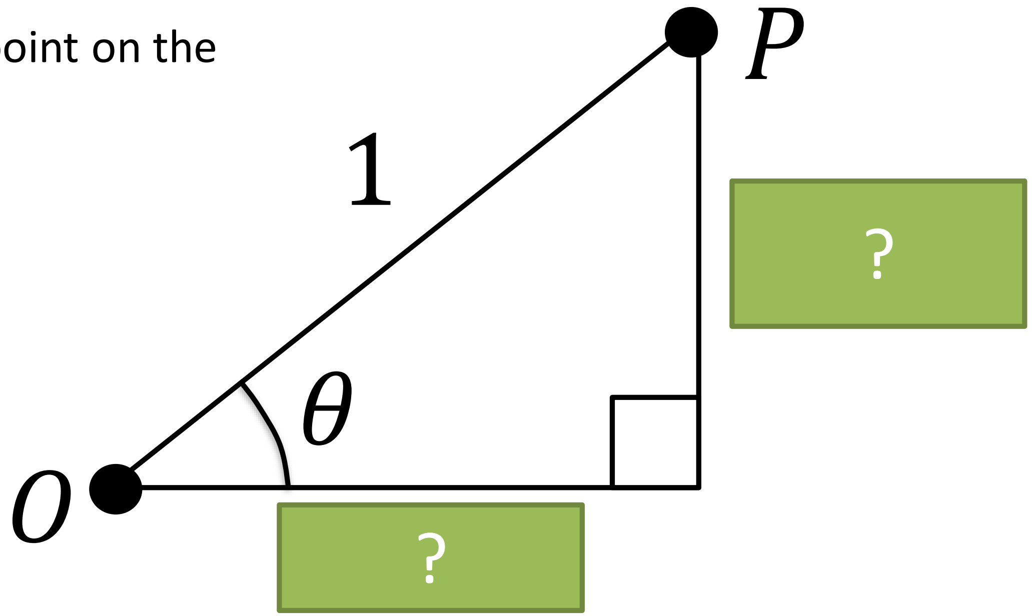
- $\sin(x) = \sin(180 - x)$
- $\cos(x) = \cos(360 - x)$
- sin, cos repeat every 360° but tan every 180°

Exercise 10A/B

Pearson Pure Mathematics Year 1/AS
Page 207, 209

Trigonometric Identities

Returning to our point on the unit circle...



1 **Then $\boldsymbol{tan\ \theta} =$** ?

2 

You are really uncool if you get this reference.

Pythagoras gives you... → ?

Application of identities #1: Proofs

Prove that $1 - \tan\theta \sin\theta \cos\theta \equiv \cos^2\theta$

$LHS =$?

$=$?

$=$?

$=$?

$$\tan x = \frac{\sin(x)}{\cos(x)}$$
$$\sin^2 x + \cos^2 x = 1$$

Recall that $\equiv$ means 'equivalent to', and just means the LHS is **always** equal to the RHS for all values of θ.
From Chapter 7 ('Proofs') we saw that usually the best method is to manipulate one side (e.g. LHS) until we get to the other (RHS).

More Examples

Edexcel C2 June 2012 Paper 1 Q16

Prove that $\tan\theta + \frac{1}{\tan\theta} \equiv \frac{1}{\sin\theta\cos\theta}$

$LHS \equiv$?

$\equiv$?

$\equiv$?

$\equiv$?

Fro Tip #2: In any addition/subtraction involving at least one fraction (with trig functions), always combine algebraically into one.

Simplify $5 - 5\sin^2\theta$

?

Fro Tip #3: Look out for $1-\sin^2\theta$ and $1-\cos^2\theta$. Students often don't spot that these can be simplified.

Test Your Understanding

Prove that $\frac{\tan x \cos x}{\sqrt{1-\cos^2 x}} \equiv 1$

?

Prove that $\frac{\cos^4 \theta - \sin^4 \theta}{\cos^2 \theta} \equiv 1 - \tan^2 \theta$

?

AQA IGCSE Further Maths Worksheet

Prove that $\tan^2 \theta \equiv \frac{1}{\cos^2 \theta} - 1$

?

Exercise 10C

Pearson Pure Mathematics Year 1/AS
Page 211-212

Extension:

[MAT 2008 1C] The simultaneous equations in x, y,

$$(\cos\theta)x - (\sin\theta)y = 2$$
$$(\cos\theta)x + (\sin\theta)y = 1$$

are solvable:

A) for all values of θ in range $0 \leq \theta < 2\pi$
B) except for one value of θ in range $0 \leq \theta < 2\pi$
C) except for two values of θ in range $0 \leq \theta < 2\pi$
D) except for three values of θ in range $0 \leq \theta < 2\pi$

?

Solving Trigonometric Equations

Remember those trigonometric angle laws (on the right) earlier this chapter? They're about to become **super freakin' useful**!

Reminder of 'trig laws':

- $\sin(x) = \sin(180 - x)$
- $\cos(x) = \cos(360 - x)$
- sin, cos repeat every $360°$ but tan every $180°$

Solve $\sin\theta = \frac{1}{2}$ in the interval $0 \leq \theta \leq 360°$.

?

Froculator Note:
When you do sin^{-1}, cos^{-1} and tan^{-1} on a calculator, it gives you only one value, known as the **principal value**.

Solve $5\tan\theta = 10$ in the interval $-180° \leq \theta < 180°$

?

Fro Tip: Look out for the solution range required. $-180 \leq \theta < 180°$ is a particularly common one.

tan repeats every $180°$, so can add/subtract $180°$ as we please.

Slightly Harder Ones…

Solve $\sin\theta = -\frac{1}{2}$ in the interval $0 \leq \theta \leq 360°$.

?

Solve $\sin\theta = \sqrt{3}\cos\theta$ in the interval $0 \leq \theta \leq 360°$.

?

Hint: The problem here is that we have two different trig functions. Is there anything we can divide both sides by so we only have one trig function?

Test Your Understanding

Solve $2\cos\theta = \sqrt{3}$ in the interval $0 \leq \theta \leq 360°$.

?

Solve $\sqrt{3}\sin\theta = \cos\theta$ in the interval $-180° \leq \theta \leq 180°$.

?

Exercise 10D

Pearson Pure Mathematics Year 1/AS
Page 215-216

Harder Equations

Harder questions replace the angle θ with a linear expression.

Solve $\cos 3x = -\frac{1}{2}$ in the interval $0 \leq x \leq 360°$.

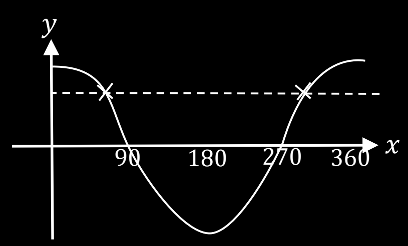
?

STEP 1: Adjust the range of values for θ to match the expression inside the cos.

STEP 2: Immediately after applying an inverse trig function (and BEFORE dividing by 3!), find all solutions up to the end of the interval.

STEP 3: Then do final manipulation to each value.

Froflections: As mentioned before, in general you tend to get a pair of values per 360° (for any of sin/cos/tan), except for $\cos\theta = \pm 1$ or $\sin\theta = \pm 1$:



Thus once getting your first pair of values (e.g. using $\sin(180 - \theta)$ or $\cos(360 - \theta)$ to get the second value), keep adding 360° to generate new pairs.

Further Examples

Solve $\sin(2x + 30°) = \frac{1}{\sqrt{2}}$ in the interval $0 \leq x \leq 360°$.

?

Solve $\sin x = 2\cos x$ in the interval $0 \leq x < 300°$

?

Test Your Understanding

Edexcel C2 Jan 2013 Q4

Solve, for $0 \leq x < 180°$,

$$\cos(3x - 10°) = -0.4,$$

giving your answers to 1 decimal place. You should show each step in your working.

(7)

?

Exercise 10E

Pearson Pure Mathematics Year 1/AS
Page 218-219

Quadratics in sin/cos/tan

We saw that an equation can be 'quadratic in' something, e.g. $x - 2\sqrt{x} + 1 = 0$ is 'quadratic in $\sqrt{x}$', meaning that $\sqrt{x}$ could be replaced with another variable, say y, to produce a quadratic equation $y^2 - 2y + 1 = 0$.

Solve $5\sin^2 x + 3\sin x - 2 = 0$ in the interval $0 \leq x \leq 360°$.

Method 1: Use a substitution.

?

Method 2: Factorise without substitution.

?

Fropinion: I'd definitely advocate Method 2 provided you feel confident with it. Method 1 feels clunky.

More Examples

Solve $\tan^2\theta = 4$ in the interval $0 \leq x \leq 360°$.

?

Solve $2\cos^2 x + 9\sin x = 3\sin^2 x$ in the interval $-180° \leq x \leq 180°$.

?

Tip: We have an identity involving sin^2 and cos^2, so it makes sense to change the squared one that would match all the others.

Test Your Understanding

Edexcel C2 Jan 2010 Q2

(*a*) Show that the equation

$$5 \sin x = 1 + 2 \cos^2 x$$

can be written in the form

$$2 \sin^2 x + 5 \sin x - 3 = 0.$$

(2)

(*b*) Solve, for $0 \leq x < 360°$,

$$2 \sin^2 x + 5 \sin x - 3 = 0.$$

(4)

?

Exercise 10F

Pearson Pure Mathematics Year 1/AS
Page 221-222

Extension

1 *[MAT 2010 1C]* In the range $0 \leq x < 360°$, the equation

$$\sin^2 x + 3\sin x \cos x + 2\cos^2 x = 0$$

Has how many solutions?

?

2 *[MAT 2014 1E]* As x varies over the real numbers, the largest value taken by the function $(4\sin^2 x + 4\cos x + 1)^2$ equals what?

?