P1 Chapter 2 :: Quadratics

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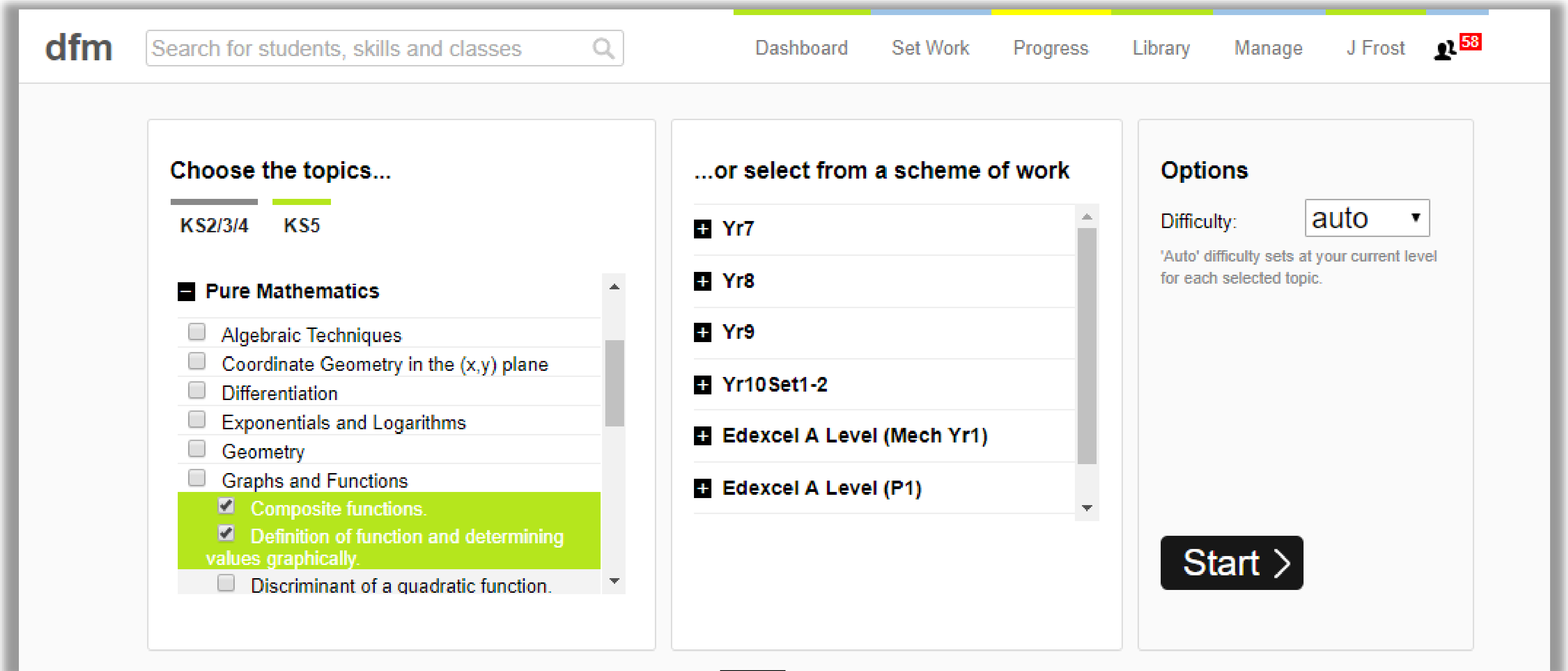

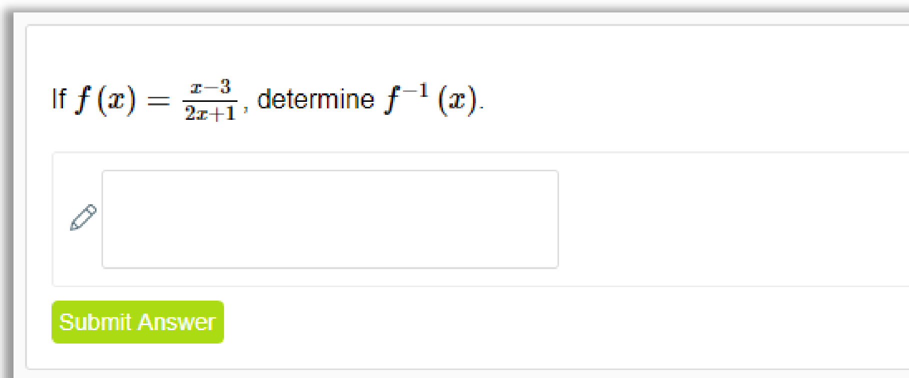

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Chapter Overview

Again, most of the content in this chapter is not new, but brings together a variety of algebraic and graph sketching GCSE skills regarding quadratic equations.

1:: Solving quadratic equations

Solve
$$(x+1)^2 - 3(x+1) + 2 = 0$$

a) Square root
b) Factorising
c) Formula
d) Completing the square

2:: Quadratics as functions

If $f(x) = x^2 + 2x$, find the roots of $f(x)$.

3:: Quadratic Graphs

Sketch $y = x^2 + 4x - 5$, indicating the coordinate of the turning point and any intercepts with the axes.

4:: The Discriminant

Find the range of values of k for which $x^2 + 4x + k = 0$ has two distinct real solutions.

5:: Modelling with Quadratics

NEW! (since GCSE)
This subtopic is completely new.

Why do we care about quadratics…

Good question!
There are a number of **practical scenarios** where a quadratic relationship arises between variables…

Summations

The sum of 1 to n is $\frac{1}{2}n^2 + \frac{1}{2}n$. I recently had to make a Sports Day spreadsheet which took the number of competing school 'houses' and worked out the total points allocated for an event by finding the formula of the resulting quadratic sequence.

1 competitor: $\mathbf{8}$ points
2 competitors: $\mathbf{8+7=15}$
3 competitors: $\mathbf{8+7+6=21}$
$\boldsymbol{n}$ competitors: $\boldsymbol{\frac{17}{2}n - \frac{1}{2}n^2}$

Projectile Motion

A projected object, acting only under gravity, follows a *parabolic* trajectory, i.e. its path can be described using a quadratic equation.
In Mechanics you will see the formula:

$$s = ut + \frac{1}{2}at^2$$

Where s is distance/displacement, u is initial speed/velocity, t is time and a is acceleration (in this case gravitational acceleration).
This equation is quadratic in t.

Why do we care about quadratics…

Hannah’s sweets

19 There are n sweets in a bag.
6 of the sweets are orange.
The rest of the sweets are yellow.

Hannah takes at random a sweet from the bag.
She eats the sweet.

Hannah then takes at random another sweet from the bag.
She eats the sweet.

The probability that Hannah eats two orange sweets is $\frac{1}{3}$

(a) Show that $n^2 - n - 90 = 0$

Quadratic expressions also regularly emerge when there’s a product of two expressions involving the same variable (in this case because Hannah ate two orange sweets).

1 :: Solving Quadratic Equations

A quadratic equation is any equation which can be written in the form $ax^2+bx+c = 0$ where a cannot equal 0.

$$x^2 + 5x = 6$$

There are five ways of solving a quadratic equation that we need to consider:

a. Square root (without factorisation)
b. Factorisation
c. Quadratic Formula
d. Completing the square
e. Graphically

1 :: Solving Quadratic Equations

a. Solving without factorising (square root)

$$(x-1)^2 = 5$$

?

If you can't see why the $\pm$ is required, think about the solutions to: $x^2 = 4$.
$2^2 = 4$, but $(-2)^2 = 4$ as well!
So $x = \pm 2$.

b. By factorisation

$$x^2 + 5x - 6 = 0$$

?

1 :: Solving Quadratic Equations

Quadratics 'in disguise'

When we have an expression like say $x^2 + 3x - 2$, we say it is "quadratic in x". Sometimes it may be difficult to recognise a quadratic equation. In trigonometry you will have to solve equations like $(\sin x)^2 + 3\sin x + 2$. We say that the expression is "quadratic in $\sin x$".

At other times we can solve a more complicated equation by "turning" it into a quadratic. If the power of one term is twice the power of another then we can often solve this way.

Either use a suitable substitution so that you have a 'normal' quadratic, or go straight for the factorisation if you're feeling more confident (recommended!).

Example: Solve $x - 6\sqrt{x} + 8 = 0$

substitution

hardcore

?

?

1 :: Solving Quadratic Equations

c. Using the quadratic formula.

If $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

?

“Use of Technology” Monkey says:
The ‘equation’ mode on your calculator will solve quadratics in the form $ax^2 + bx + c = 0$. When you’re asked for the ‘order’, use 2 (we’ll see why later).

Test Your Understanding

1 Solve $(x+3)^2 = x+5$ using factorisation.

?

2 Solve $(2x+1)^2 = 5$

?

3 Solve $\sqrt{x+3} = x-3$

?

4 Solve $2x + \sqrt{x} - 1 = 0$

?

Exercise 2A/2B

Pearson Pure Mathematics Year 1/AS
Pages 20-22

Extension: (Full Database: http://www.drfrostmaths.com/resources/resource.php?rid=268)

1 **(i)** Use the substitution $\sqrt{x} = y$ (where $y \geqslant 0$) to find the real root of the equation

$$x + 3\sqrt{x} - \tfrac{1}{2} = 0\,.$$

(ii) Find all real roots of the following equations:

(a) $x + 10\sqrt{x+2} - 22 = 0$;

(b) $x^2 - 4x + \sqrt{2x^2 - 8x - 3} - 9 = 0$.

? ii(a)

? ii(b)

? i

1d. Completing the Square

"Completing the square" means putting a quadratic in the form $(x+a)^2+b$ or $a(x+b)^2+c$

The **underlying reason we do this is because x only appears once in the expression** (e.g. in $(x+2)^2+3$ vs x^2+4x+7), which makes it algebraically easier to handle. This has a number of consequences:

a. Solving Quadratics

If we have a completed square:

$$(x+4)^2-7=0$$

we saw at the start of the chapter how we could rearrange to make x the subject.

Indeed using the quadratic formula is actually solving the quadratic by completing the square – it's just someone has done the work for us already!

b. Sketching Quadratics

We'll see later that if $y=(x+a)^2+b$, then the minimum point is $(-a,b)$

c. In integration

In Further Maths, completing the square allows us to 'integrate' expressions like:

$$\int \frac{1}{x^2-4x+5}\,dx$$

(you will cover integration later this module)

Completing the Square (Recap: IGCSE)

1. Working with algebraic EXPRESSIONS

Example:

Write $x^2 + 12x$ in completed square form:

Step 1: Consider the coefficient of x 12

Step 2: Halve its value, add to x and square the bracket $(x+6)^2$

Step 3: Imagine expanding this bracket, what do we end up with that we don't actually want? +36

Step 4: Adjust to bracket to compensate. $(x+6)^2$ - 36

Since $(x+6)^2 = x^2 + 12x + 36$, we want to discard the 36, so 'throw it away' by subtracting.

$$x^2 + 12x = (x+6)^2 - 36$$

Textbook Note:
The textbook uses the formula

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

and similarly

$$ax^2 + bx + c$$
$$= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

My personal judgement is that it's <u>not</u> worth memorising these and you should instead think through the steps. Even the textbook agrees!

Completing the Square (Recap: IGCSE)

Further Examples:

1. $x^2 + 8x =$?
2. $.\,x^2 - 2x =$?
3. $x^2 - 6x + 7 =$?

Notice that despite the a being negative, we still subtract after the bracket as $(-1)^2$ is positive.

Completing the Square

Expressions with "a" not equal to 1: We cannot simplify divide each term by a value because we are working with expressions not equations.

Express $2x^2 + 12x + 7$ in the form $a(x+b)^2 + c$

= ?

= ?

= ?

= ?

Factorise out coefficient of x^2. You can leave the constant term outside the bracket.

Complete the square inside the bracket (you should have two sets of brackets)

Expand out outer bracket.

Express $5 - 3x^2 + 6x$ in the form $a - b(x+c)^2$

= ?

= ?

= ?

= ?

= ?

It may help to write in the form $ax^2 + bx + c$ first.

Test Your Understanding

Express $3x^2 - 18x + 4$ in the form $a(x+b)^2 + c$

$=$

$=$

$=$

$=$

?

Express $20x - 5x^2 + 3$ in the form $a - b(x+c)^2$

$=$

$=$

$=$

$=$

$=$

?

Solving by Completing the Square

2. Working with algebraic EQUATIONS

Solve the equation:

$$3x^2 - 18x + 4 = 0$$

? First step

? And the rest…

Note: Previously we factorised out the 3. This is because $3x^2 - 18x + 4$ on its own is an **expression,** so dividing by 3 (instead of factorising) would change the expression.

However, in an equation, we can divide both sides by 3 without affecting the solutions.

Exercise 2C/2D

Pearson Pure Mathematics Year 1/AS
Page 23-24

Proving the Quadratic Formula

If $ax^2 + bx + c = 0$, prove that $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$ax^2 + bx + c = 0$$

?

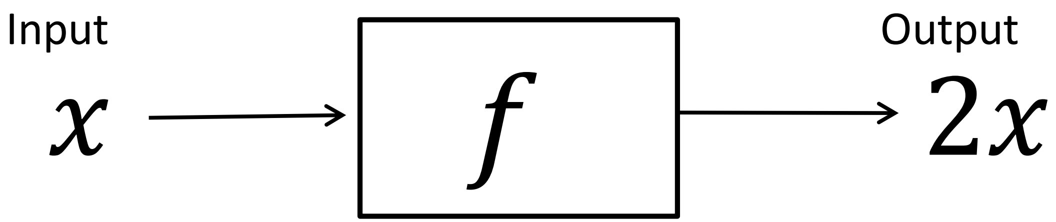
Just use exactly the same method as you usually would!

!

3 :: Functions

A function is something which **provides a rule on how to map inputs to outputs**.

We saw at GCSE that functions were a formal way of describing a ‘number machine’:



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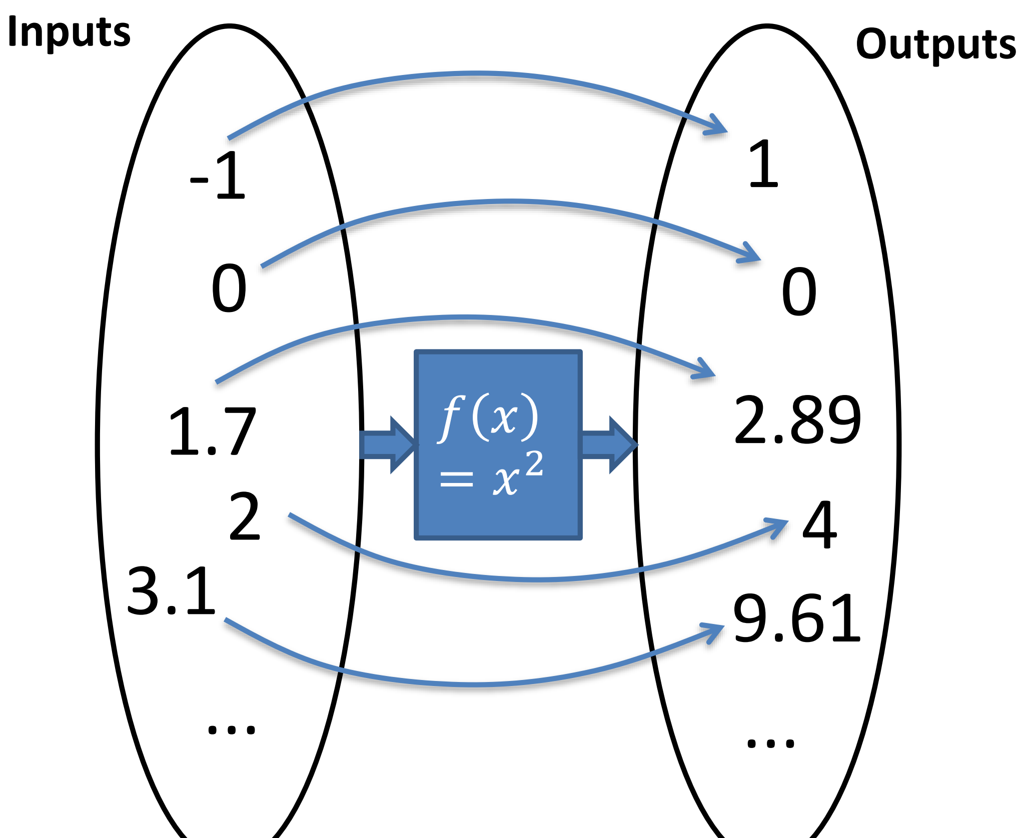
3 :: Functions

You'll cover functions extensively in future chapters, but for now, you need to understand the following concepts:



The domain of a function could potentially be **any** real number. If so, we'd write:

?

We might be interested in what inputs x give an output of 0. These are known as the ? of the function.

✏ The **domain** of a function is the set of possible inputs.

✏ The **range** of a function is the set of possible outputs.

✏ The **roots/zeroes** of a function are the values of x for which $f(x) = 0$.

Examples

If $f(x) = x^2 - 3x$ and $g(x) = x + 5, \quad x \in \mathbb{R}$

a) Find $f(-4)$

b) Find the values of x for which $f(x) = g(x)$

c) Find the roots of $f(x)$.

d) Find the roots of $g(x)$.

Fro Note: The domain is usually stated for you.

a

?

b

?

c

?

d

?

Examples

Determine the minimum value of the function $f(x) = x^2 - 6x + 2$, and state the value of x for which this minimum occurs.

This means we want to minimise the **output** of the function.

You might try a (clumsy) approach of trying a few values of x and try to see what makes the output as small as possible...

?

But the best way to find the minimum/maximum value of a quadratic is to **complete the square**:

?

Quickfire Questions

$f(x)$	Completed square	Min/max value of $f(x)$	x for which this min/max occurs
x^2+4x+9	?	?	?
$x^2-10x+21$	?	?	?
$10-x^2$	?	?	?
$8-x^2+6x$	?	?	?

Test Your Understanding

1 Find the minimum value of $f(x) = 2x^2 + 12x - 5$ and state the value of x for which this occurs.

?

2 Find the roots of the function $f(x) = 2x^2 + 3x + 1$

?

3 Find the roots of the function $f(x) = x^4 - x^2 - 6$

?

Exercise 2E

Pearson Pure Mathematics Year 1/AS
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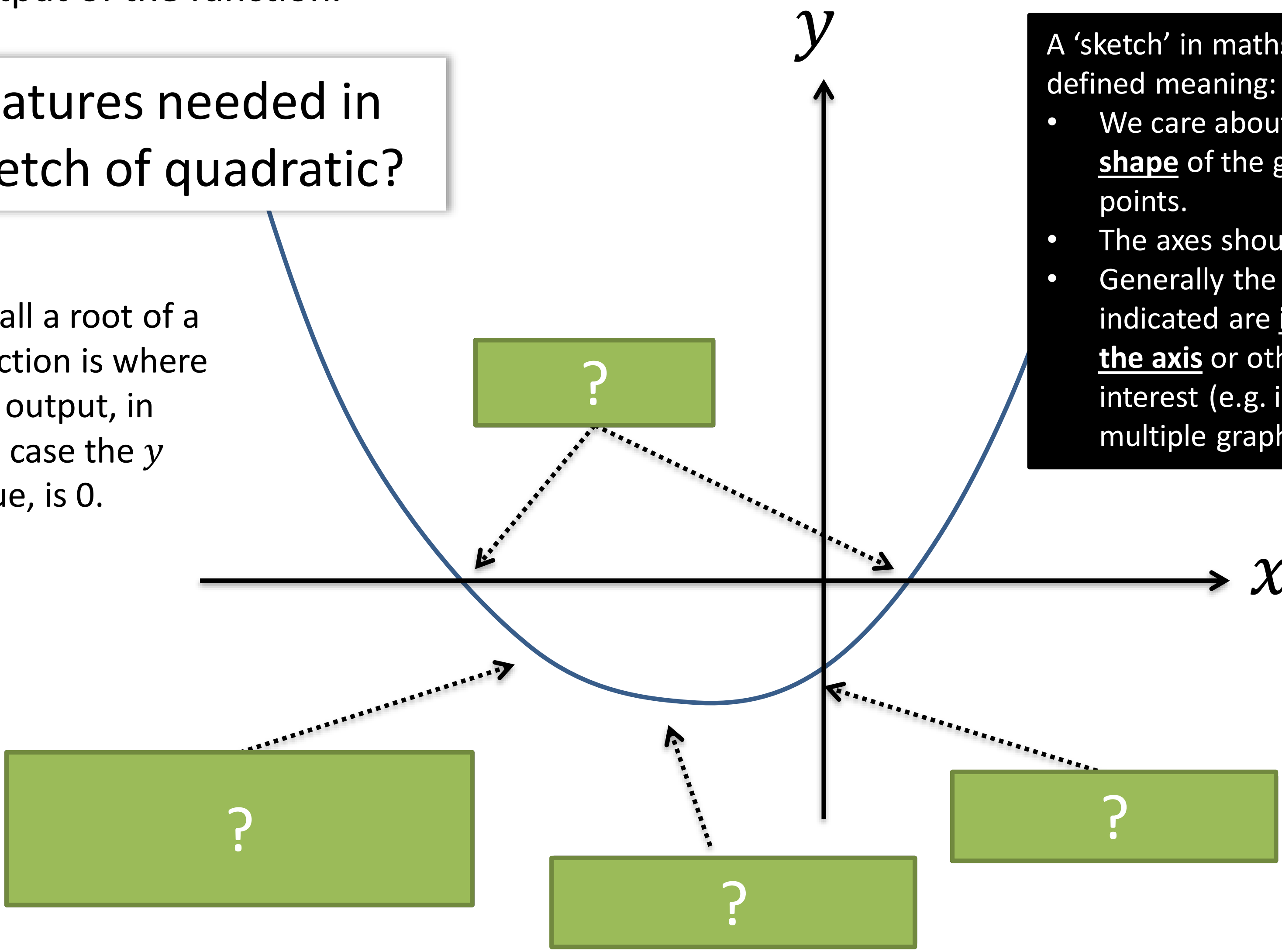
5. e :: Quadratic Graphs

Recall that x refers to the input of a function, and the expression $f(x)$ refers to the output. For graph sketches, we often write $y = f(x)$, i.e. we set the y values to be the output of the function.

Features needed in sketch of quadratic?

Recall a root of a function is where the output, in this case the y value, is 0.



A ‘sketch’ in maths has a clearly defined meaning:

- We care about the **general shape** of the graph, not exact points.
- The axes should have **no scale**.
- Generally the only coordinates indicated are **intercepts with the axis** or other points of interest (e.g. intersections of multiple graphs)

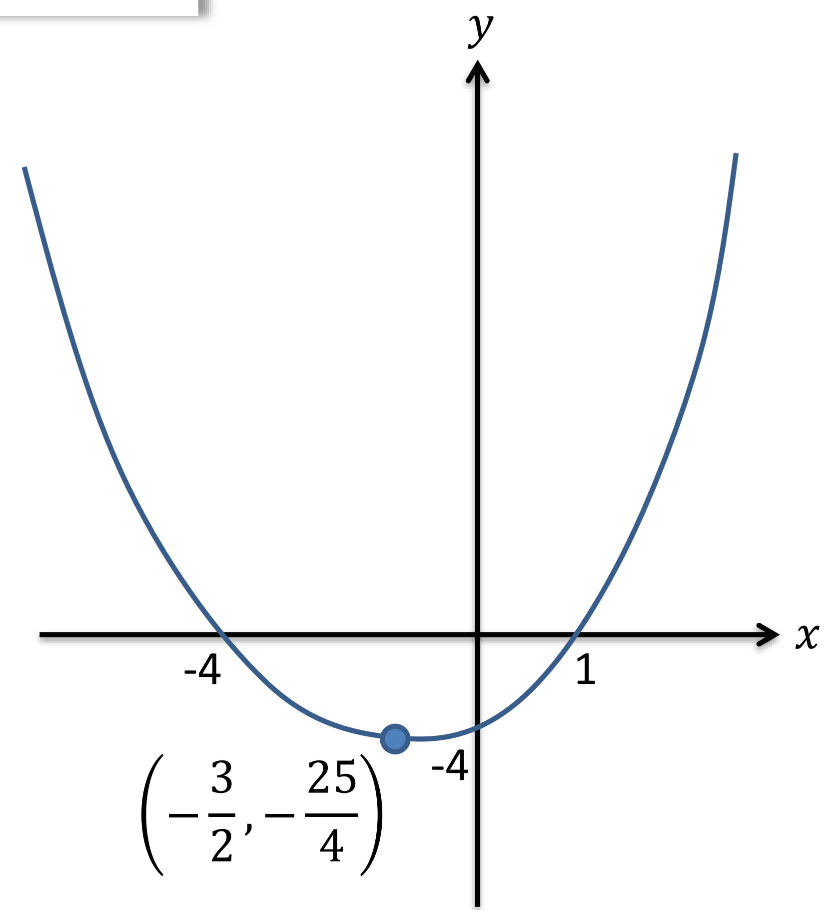
Example

Sketch the graph of $y = x^2 + 3x - 4$ and find the coordinates of the turning point.

Roots: ?

y-intercept: ?

Turning point: ?



Recall that if $f(x) = (x+a)^2 + b$, the minimum output is b and $-a$ is the x value which minimises it. i.e. Turning point is $(-a, b)$

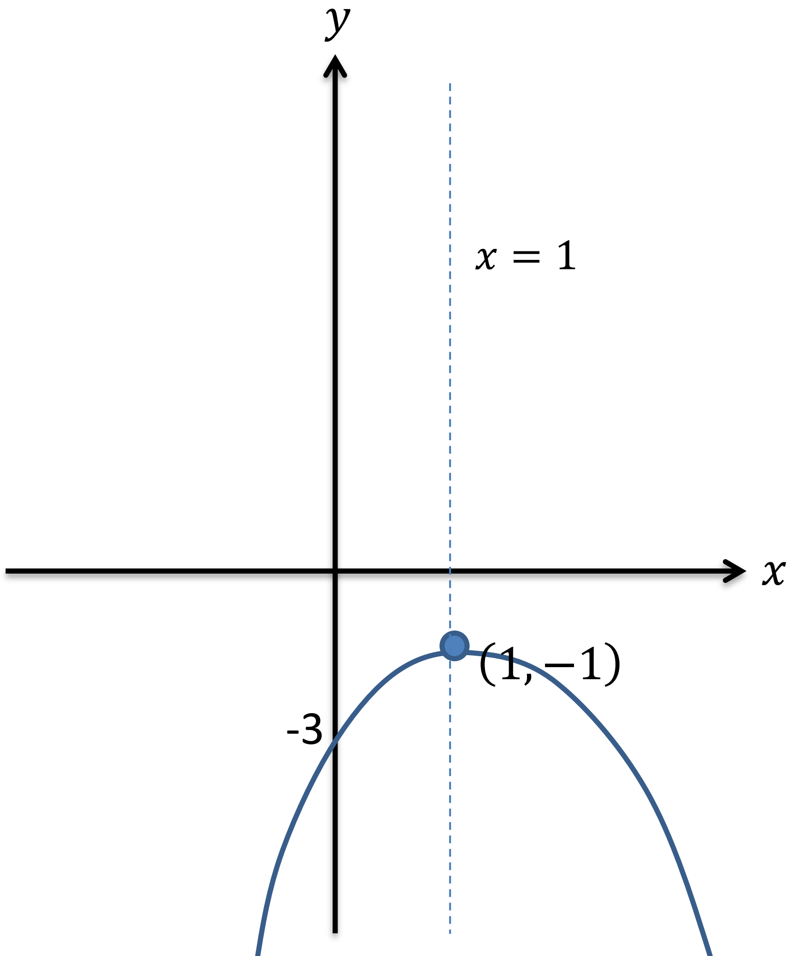
Example

Sketch the graph of $y = 4x - 2x^2 - 3$ and find the coordinates of the turning point. Write down the equation of the line of symmetry.

Roots: ?

y-intercept: ?

Turning point: ?



Line of reflection: ?

Test Your Understanding

Sketch the following, indicating any intercepts with the axis, the turning point and the equation of the line of symmetry.

a $y = x^2 + 4$

?

c $y = 5x + 3 - 2x^2$

?

b $y = x^2 - 7x + 10$

?

d $y = x^2 + 4x + 11$

?

Exercise 2F

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Extension Question:

[MAT 2003 1H] Into how many regions is the plane divided when the following three parabolas are drawn?

$$y = x^2$$
$$y = x^2 - 2x$$
$$y = x^2 + 2x + 2$$

?

Starter

How many **distinct** real solutions do each of the following have?

$$x^2 - 12x + 36 = 0$$

?

$$x^2 + x + 3 = 0$$

?

$$x^2 - 2x - 1 = 0$$

?

5 :: The Discriminant

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: Roots of a **function** f are the values of x such that $f(x) = 0$.
Similarly the roots of an **equation** are solutions to an equation in the form $f(x) = 0$

Looking at this formula, when in general do you think we have:

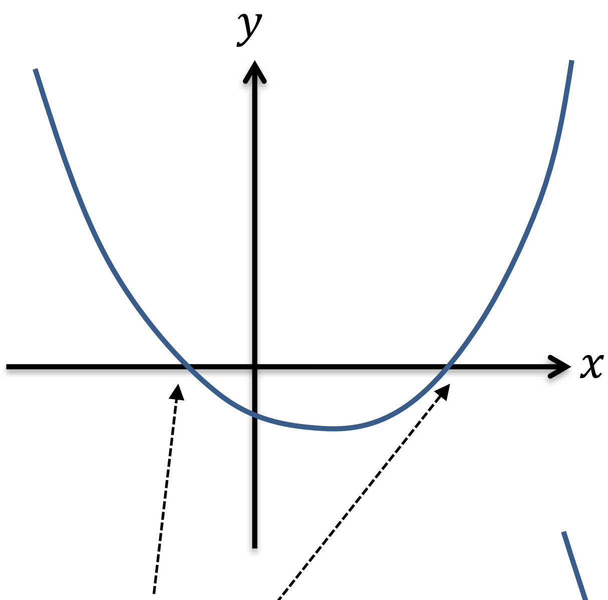
- No real roots?
- Equal roots?
- Two distinct roots?

?
?
?

Because adding 0 or subtracting 0 in the quadratic formula gives the same value.

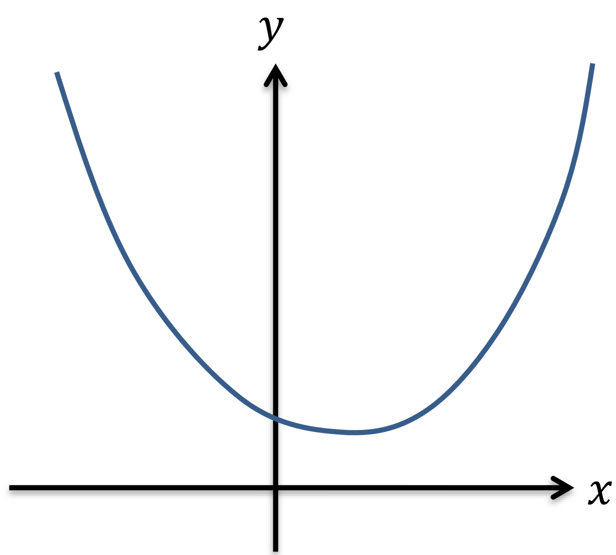
$b^2 - 4ac$ is known as the discriminant.

5 :: The Discriminant



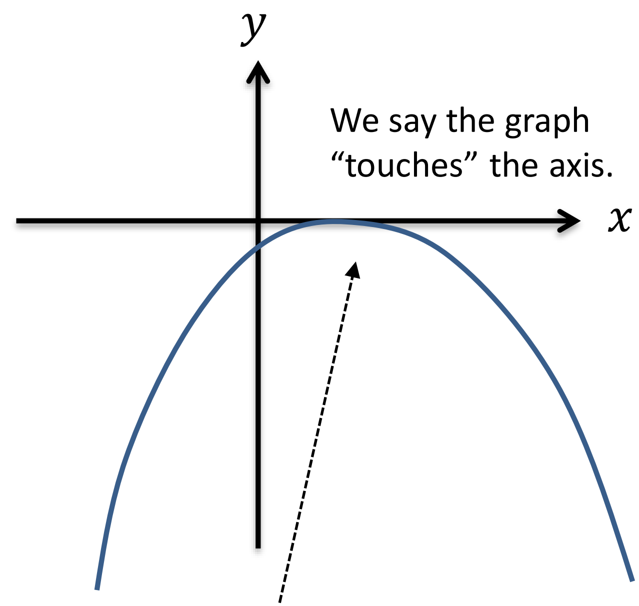
Distinct real roots

$$b^2 - 4ac > 0$$



No real roots

$$b^2 - 4ac < 0$$

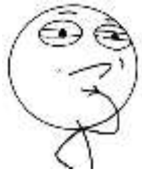


Equal roots

$$b^2 - 4ac = 0$$

Just for your interest…

Why do we say "Equal Roots" not "One root"?



$$x^2 - 12x + 36 = 0$$

Using the quadratic formula gives us the same value in both $+$ and $-$ cases: $x = 6$.

You might wonder why we say "*it has one repeated root*" or "*it has equal roots*", i.e. indicating we have **2 roots** (but with the same value). Why not say it has 1 root?

It is due to the **Fundamental Theorem of Algebra**:

"Every polynomial of order n has exactly n roots."

A **polynomial** is an expression with non-negative integer powers of x, i.e. $a + bx + cx^2 + dx^3 + \cdots$ All linear, quadratic and cubic expressions are examples of polynomials.

The **order** of a polynomial is its highest power of x. So the order of a quadratic is 2, and a cubic 3.

These roots might be repeated or might not be 'real' roots. $\sqrt{-1}$ is known as a **complex number**, which you will encounter if you do FM. But **it is still a value**!

The theorem means that a quadratic (order 2) will **always** have 2 roots. This is why you should say "no **real** roots" when $b^2 - 4ac < 0$ rather than "no roots", because there are still roots – it's just they're not 'real'! Similarly we must say "*equal roots*" because there are still 2 roots.

Despite the theorem being a simple statement, it was only until 1806 that it was first proven by **Argand**. Clearly by using the quadratic formula we can show a quadratic equation has 2 roots. We can use similar formulas to show that a cubic has 3 roots and a quartic 4 roots. But there is **provably** no such formulae for order 5 (quintics) and beyond. So we have to prove for example that 5 roots exist for a quintic, despite us having no way to find these exact roots!

One side result of the Fundamental Theorem of Algebra is that every polynomial can be written as a product of linear and/or quadratic expressions.

Leibniz claimed in 1702 that a polynomial of the form $x^4 + a^4$ cannot be written in this way. He then got completely burned by Euler in 1742 who managed to do so:

$$x^4 + a^4 = \left(x^2 + a\sqrt{2}\,x + a^2\right)\left(x^2 - a\sqrt{2}\,x + a^2\right)$$

There are various other 'Fundamental Laws'. The '**FL of Arithmetic**' you encountered at KS3, which states that "*every positive integer* > 1 *can be written as a product of primes in one way only*". You will encounter the '**FL of Calculus**' in Chapter 13.

Quickfire Questions

Equation	Discriminant	Number of Distinct Real Roots
$x^2 + 3x + 4 = 0$	?	?
$x^2 - 4x + 1 = 0$	?	?
$x^2 - 4x + 4 = 0$	?	?
$2x^2 - 6x - 3 = 0$	?	?
$x - 4 - 3x^2 = 0$	?	?
$1 - x^2 = 0$	?	?

Problems involving the discriminant

8. The equation $x^2 + 2px + (3p + 4) = 0$, where p is a positive constant, has equal roots.

(*a*) Find the value of p.

(4)

(*b*) For this value of p, solve the equation $x^2 + 2px + (3p + 4) = 0$.

(2)

a) $a =$? $b =$? $c =$?

?

b) ?

Fro Tip: Always start by writing out a, b and c explicitly.

Test Your Understanding

$$x^2 + 5kx + (10k + 5) = 0$$

where k is a positive constant.

Given that this equation has equal roots, determine the value of k.

?

Find the range of values of k for which $x^2 + 6x + k = 0$ has two distinct real solutions.

?

Exercise 2G

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Extension Questions:

1 [MAT 2009 1C] Given a real constant c, the equation

$$x^4 = (x - c)^2$$

Has four real solutions (including possible repeated roots) for:

A) $c \leq \frac{1}{4}$
B) $-\frac{1}{4} \leq c \leq \frac{1}{4}$
C) $c \leq -\frac{1}{4}$
D) all values of c

?

2 [MAT 2006 1B] The equation $(2 + x - x^2)^2 = 16$ has how many real root(s)?

?

3 [MAT 2011 1B] A rectangle has perimeter P and area A. The values P and A must satisfy:

A) $P^3 > A$
B) $A^2 > 2P + 1$
C) $P^2 \geq 16A$
D) $PA > A + P$

?

Modelling

The new A Level has a particular emphasis of the application of theory to real-life situations. A mathematical model is the maths used to model such a situation, possibly with some simplifying assumptions.

Example (from textbook): A spear is thrown over level ground from the top of a tower. The height, in metres, of the spear above the ground after t seconds is modelled by the function: $h(t) = 12.25 + 14.7t - 4.9t^2, \quad t \geq 0$

a) Interpret the meaning of the constant term 12.25 in the model.
b) After how many seconds does the spear hit the ground?
c) Write $h(t)$ in the form $A - B(t - C)^2$, where A, B and C are constants to be found.
d) Using your answer to part c or otherwise, find the maximum height of the spear above the ground, and the time at which this maximum height is reached?

a ?

b ?

c ?

d ?

Modelling: Test Your Understanding

A rectangular car park has a perimeter of 184 metres, and the diagonal of the car park measures 68 metres.

(i) By labelling the length of the car park as x metres, formulate an equation and check that $x = 32$ satisfies the equation.

Hence find the dimensions of the car park.

(ii) Sketch the graph of the quadratic expression in part (i), and interpret each intersection with the x-axis in terms of the car park.

Modelling: Test Your Understanding

?

Exercise 2H

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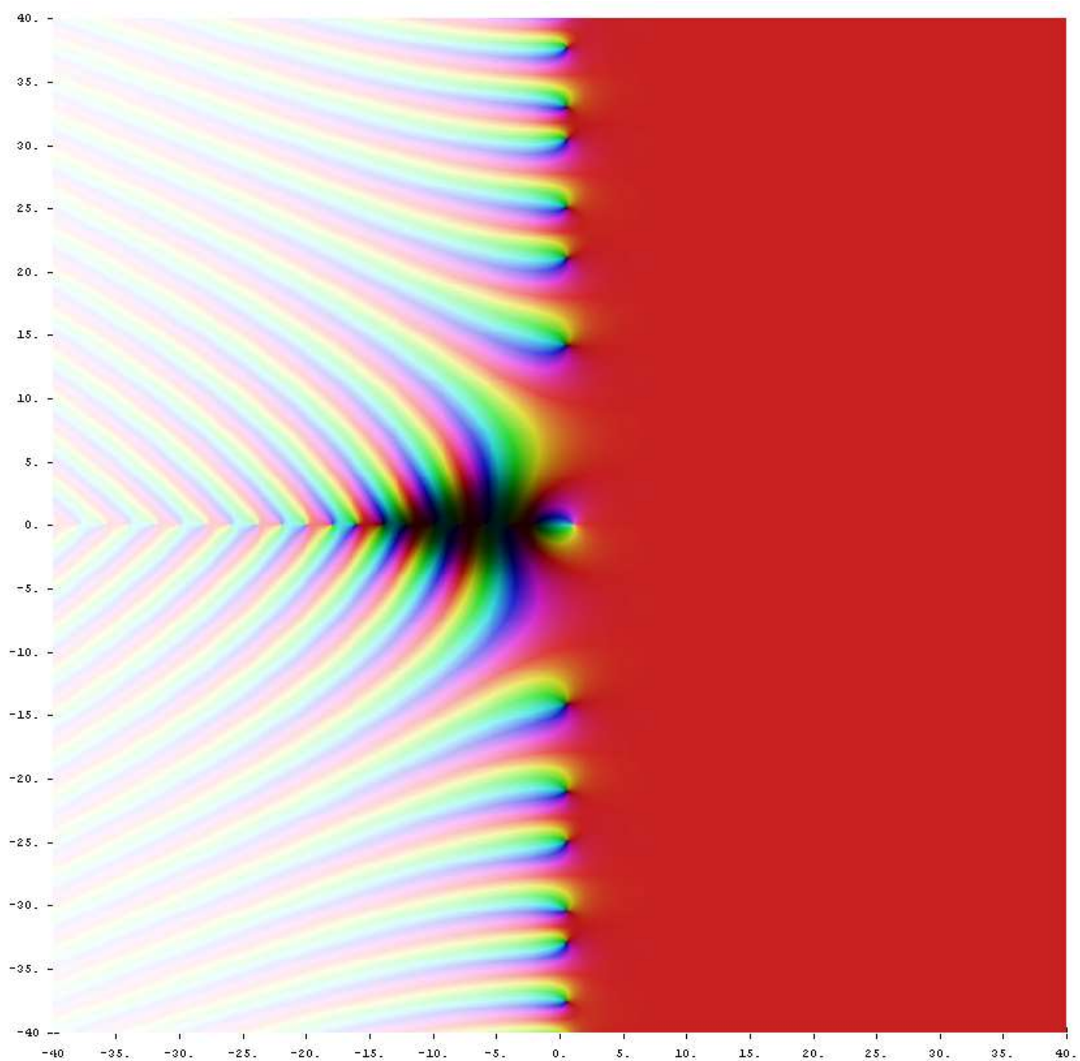
Would you like $1,000,000 for finding roots?

We saw earlier that the roots of a function f are the values x such that $f(x) = 0$.



The **Riemann Zeta Function** is a function that allows you to do the infinite sum of powers of reciprocals, e.g.

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$
$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$$

One of the 8 '**Clay Millennium Problems**' (for which solving any attracts a $1,000,000 prize) is to **showing all roots of this function have some particular from**, i.e. the form of x such that $\zeta(x) = 0$.