U6 Chapter 4

Binomial Expansion

Chapter Overview

1. Binomial Series Recap
2. Binomial Expansion for negative/fractional powers
3. Constant is not 1: $(a+b)^n$
4. Using Partial Fractions

4 **Sequences and series**	4.1	**Understand and use the binomial expansion of $(a+bx)^n$ for positive integer n; the notations $n!$ and ${}^nC_r$ link to binomial probabilities.**	**Use of Pascal's triangle.** **Relation between binomial coefficients.** **Also be aware of alternative notations such as $\binom{n}{r}$ and ${}^nC_r$** **Considered further in Paper 3 Section 4.1.**
		Extend to any rational n, including its use for approximation; be aware that the expansion is valid for $\left\lvert\frac{bx}{a}\right\rvert < 1$ (proof not required)	May be used with the expansion of rational functions by decomposition into partial fractions May be asked to comment on the range of validity.

The Binomial Series: Recap

Recall that if n is a positive integer

$$(a+b)^n = a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}b^2 + \cdots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + {}^nC_r\, x^n$$

Also $$(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n$$

Examples

1. Expand $(1+x)^{11}$ up to and including the term in x^3

2. Expand $(1-2x)^8$ up to and including the term in x^3

Binomial Expansion for Negative/ Fractional Powers

Example

1. Use the binomial expansion to find the first four terms of $\frac{1}{1+x}$

2. Use the binomial expansion to find the first four terms of $\sqrt{1-3x}$

An infinite expansion $(1+x)^n$ is valid if $|x| < 1$

Quickfire Examples:

1. Expansion of $(1+2x)^{-1}$ valid if:

2. Expansion of $(1-x)^{-2}$ valid if:

3. Expansion of $\left(1+\frac{1}{4}x\right)^{\frac{1}{2}}$ valid if:

4. Expansion of $\left(1-\frac{2}{3}x\right)^{-1}$ valid if:

Combining Expansions

(*a*) Use the binomial expansion to show that

$$\sqrt{\left(\frac{1+x}{1-x}\right)} \approx 1+x+\frac{1}{2}x^2, \qquad |x|<1$$

(6)

Test Your Understanding

1. Find the binomial expansion of $\frac{1}{(1+4x)^2}$ up to an including the term in x^3.
State the values of x for which the expansion is valid.

2.

(a) Find the binomial expansion of

$$\sqrt{(1-8x)}, \qquad |x| < \frac{1}{8},$$

in ascending powers of x up to and including the term in x^3, simplifying each term.

(6)

(b) Show that, when $x = \frac{1}{100}$, the exact value of $\sqrt{(1-8x)}$ is $\frac{\sqrt{23}}{5}$.

(2)

(c) Substitute $x = \frac{1}{100}$ into the binomial expansion in part (a) and hence obtain an approximation to $\sqrt{23}$. Give your answer to 5 decimal places.

(3)

Extension

[STEP I 2011 Q6] Use the binomial expansion to show that the coefficient of x^r in the expansion of $(1-x)^{-3}$ is $\frac{1}{2}(r+1)(r+2)$.

(i) Show that the coefficient of x^r in the expansion of $\frac{1-x+2x^2}{(1-x)^3}$ is r^2+1 and hence find the sum of the series

$$1+\frac{2}{2}+\frac{5}{4}+\frac{10}{8}+\frac{17}{16}+\frac{26}{32}+\frac{37}{64}+\cdots$$

(ii) Find the sum of the series

$$1+2+\frac{9}{4}+2+\frac{25}{16}+\frac{9}{8}+\frac{49}{64}$$

Exercise 4A Page 96-97

Dealing with $(a+b)^n$

Remember $$(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n$$

Examples

1. Find first four terms in the binomial expansion of $\sqrt{4+x}$. State the values of x for which the expansion is valid.

Quickfire First Step

What would be the first step in finding the Binomial expansion of each of these?

	First Step…	Valid when?
1. $(2+x)^{-3}$		
2. $(9+2x)^{\frac{1}{2}}$		
3. $(8-x)^{\frac{1}{3}}$		
4. $(5-2x)^{-3}$		
5. $(16+3x)^{-\frac{1}{2}}$		

Test Your Understanding

(a) Find the binomial expansion of

$$\sqrt{(9 + 8x)}, \quad |x| < \frac{9}{8}$$

in ascending powers of x, up to and including the term in x^2.
Give each coefficient as a simplified fraction.

(5)

(b) Use your expansion to estimate the value of $\sqrt{(11)}$, giving your answer as a single fraction.

(3)

Extension

[AEA 2006 Q1]

(a) For $|y| < 1$, write down the binomial series expansion of $(1-y)^{-2}$ in ascending powers of y up to and including the term in y^3.

(b) Hence, or otherwise, show that

$$1 + \frac{2x}{1+x} + \frac{3x^2}{(1+x)^2} + \cdots + \frac{rx^{r-1}}{(1+x)^{r-1}} + \cdots$$

can be written in the form $(a+x)^n$. Write down the values of the integers a and n.

(c) Find the set of values of x for which the series in part (b) is convergent.

Exercise 4B Page 99

<u>Using Partial Fractions</u>

Example

1.

a) Express $\frac{4-5x}{(1+x)(2-x)}$ as partial fractions.

b) Hence show that the cubic approximation of $\frac{4-5x}{(1+x)(2-x)}$ is $2-\frac{7}{2}x+\frac{11}{4}x^2-\frac{25}{8}x^3$

c) State the range of values of x for which the expansion is valid.

Test Your Understanding

[C4 June 2010 Q5]

10.
$$\frac{2x^2+5x-10}{(x-1)(x+2)} \equiv A + \frac{B}{x-1} + \frac{C}{x+2}.$$

(*a*) Find the values of the constants A, B and C. **(4)**

(*b*) Hence, or otherwise, expand $\dfrac{2x^2+5x-10}{(x-1)(x+2)}$ in ascending powers of x, as far as the term in x^2. Give each coefficient as a simplified fraction. **(7)**

Exercise 4C Page 102