P2 Chapter 2 :: Functions & Graphs

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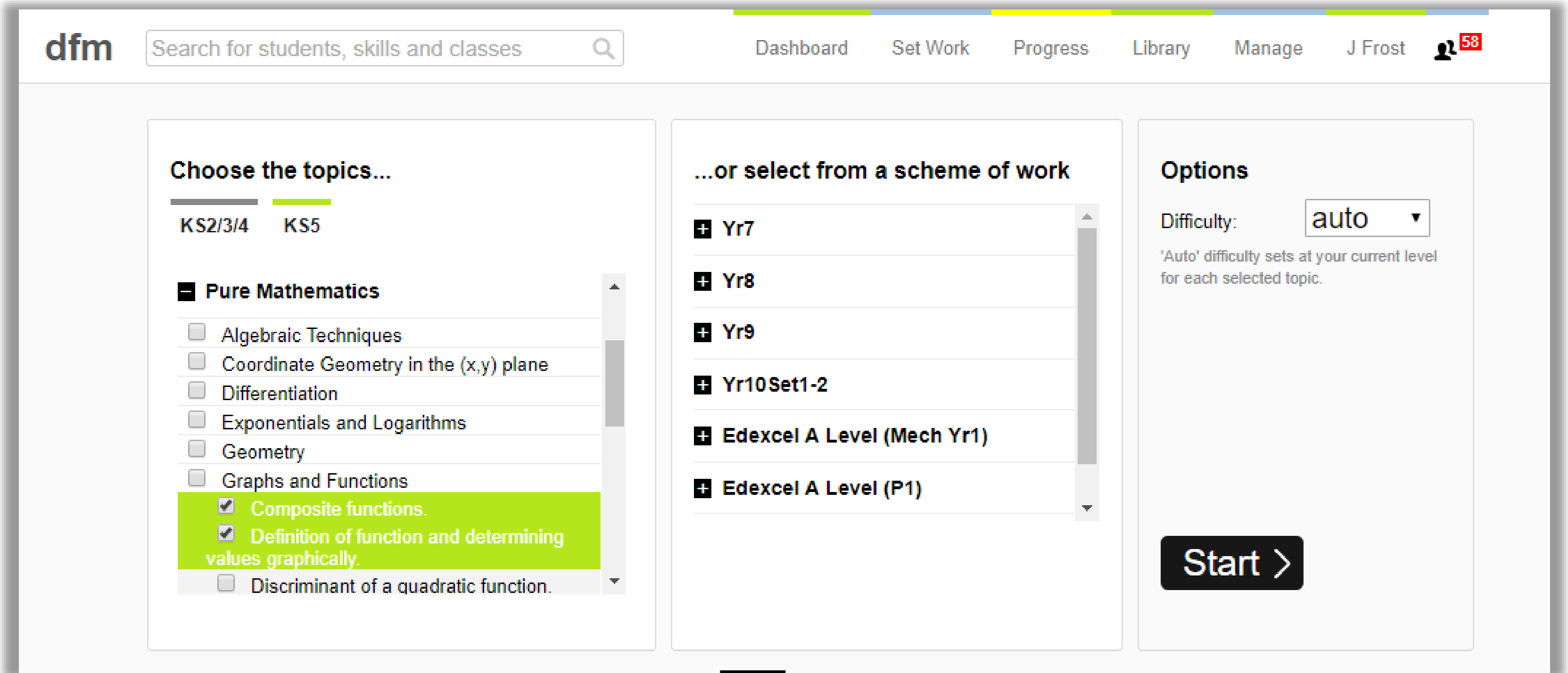

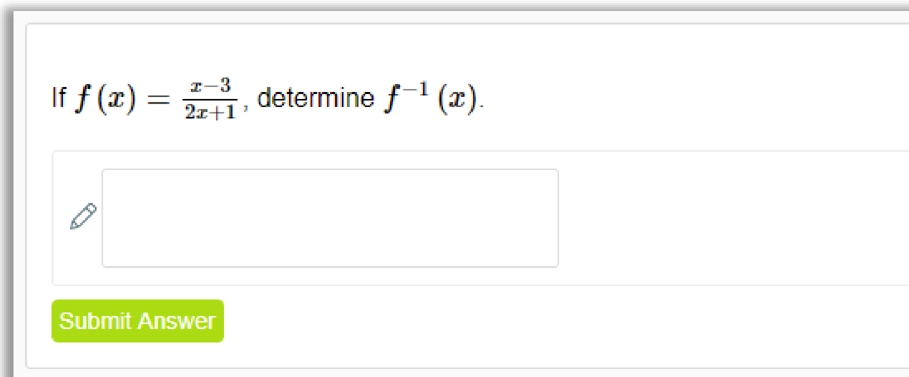

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Chapter Overview

1:: The Modulus Function

Solve $|3x - 5| = 2 - \frac{1}{2}x$

2:: Mappings vs Functions, Domain and Range

If $f(x) = x^2 - 4x + 3$, find the range of f.

3:: Composite Functions

If $f(x) = 2x + 1$ and $g(x) = x^2$, determine:

a) $fg(x)$

b) $gf(x)$

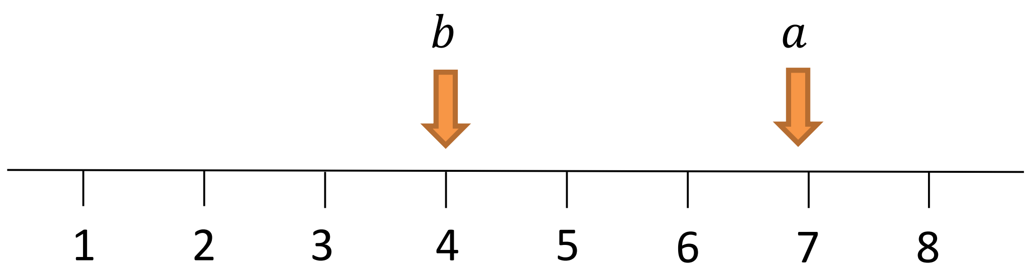
4:: Inverse Functions

If $f(x) = \frac{x+1}{2x-1}$, determine $f^{-1}(x)$.

5:: Transformations of the form $y = |f(x)|$ or $y = f(|x|)$. Combined transformations and transforming the modulus function.

1 :: The Modulus Function

The modulus of a number a, written $|a|$, is its **non-negative** numerical value.
e.g. $|6| = 6$ and $|-7.1| = 7.1$



The modulus function is particularly useful in expressing a **difference**. We generally like to quote differences as positive values, but $b - a$ may be negative if b is smaller than a. By using $|b - a|$, we get round this problem!

More fundamentally, the modulus of a value gives us its '**magnitude**', i.e. size; from Mechanics, you should also be used to the notion the distances and speeds are quoted as positive values.
And in Pure Year 1 we saw the same notation used for vectors: $|\boldsymbol{a}|$ gives us the magnitude/length of the vector $\boldsymbol{a}$. It's the same function!

Examples

If $f(x) = |2x - 3| + 1$, find

a) $f(5)$

b) $f(-2)$

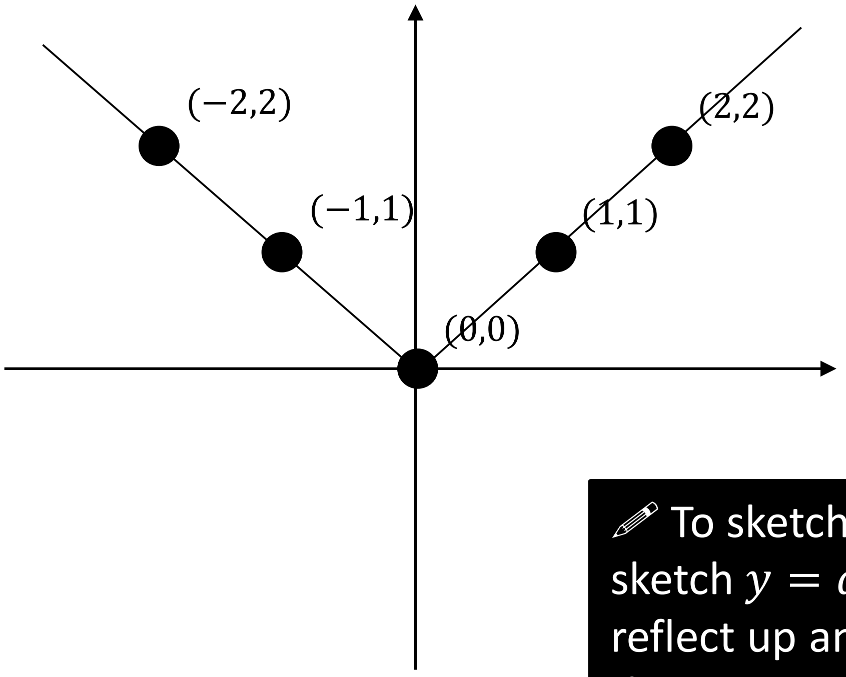
c) $f(1)$

?

Modulus Graphs

$y = |x|$

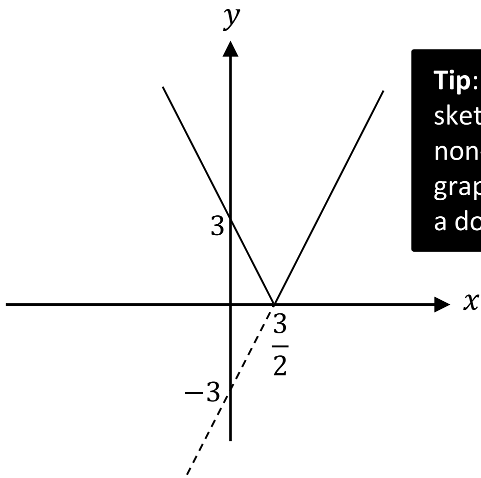
x	-2	-1	0	1	2
y	?	?	?	?	?



✏ To sketch $y = |ax + b|$, sketch $y = ax + b$ then reflect up any section below the x-axis.

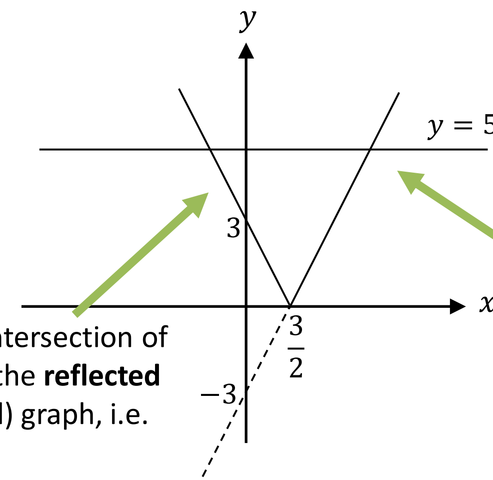
Modulus Graphs

Sketch $y = |2x - 3|$



Tip: I like to sketch the non-modulus graph first with a dotted line.

Solve $|2x - 3| = 5$



As you would have done in Pure Year 1, sketch a line for each side of the equation, so that we can use the points of intersection.

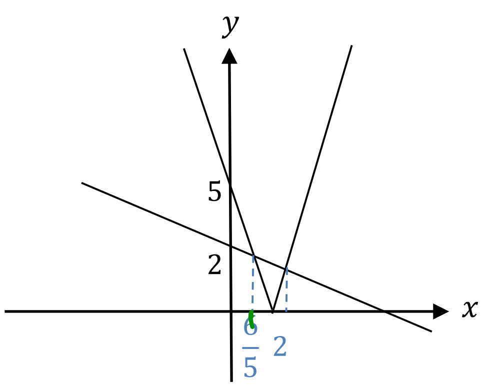
This is the intersection of $y = 5$ with the **reflected** (i.e. negated) graph, i.e. $y = 3 - 2x$.

$$\boldsymbol{3 - 2x = 5}$$
$$\boldsymbol{x = -1}$$

This is the intersection of $y = 5$ with the original **unreflected** graph, i.e. $y = 2x - 3$.

$$\boldsymbol{2x - 3 = 5}$$
$$\boldsymbol{x = 4}$$

Solve $|3x - 5| = 2 - \frac{1}{2}x$



$$\boldsymbol{3x - 5 = 2 - \frac{1}{2}x}$$
$$\boldsymbol{x = 2}$$

$$\boldsymbol{5 - 3x = 2 - \frac{1}{2}x}$$
$$\boldsymbol{x = \frac{6}{5}}$$

Solve $|3x - 5| > 2 - \frac{1}{2}x$

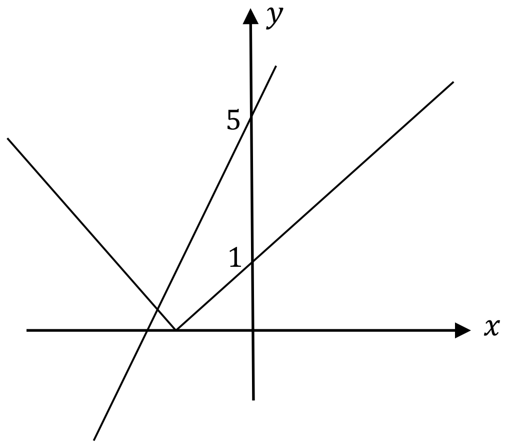
The graph of $y = |3x - 5|$ needs to be above $y = 2 - \frac{1}{2}x$.

By observation (and using our points of intersection), this occurs when

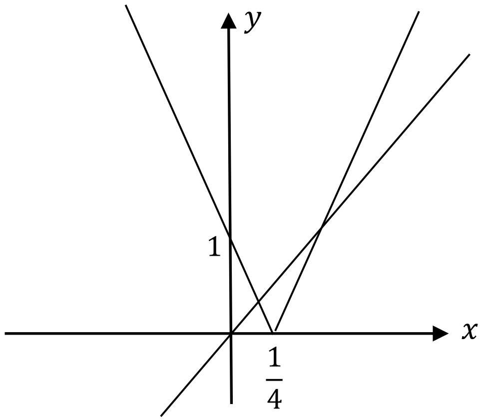
$$x < \frac{6}{5} \qquad x > 2$$

Test Your Understanding

Solve $|x+1| = 2x+5$
(be careful – there's only one solution!)



Solve $|4x-1| < 2x$



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Exercise 2A

Pearson Pure Mathematics Year 2/AS
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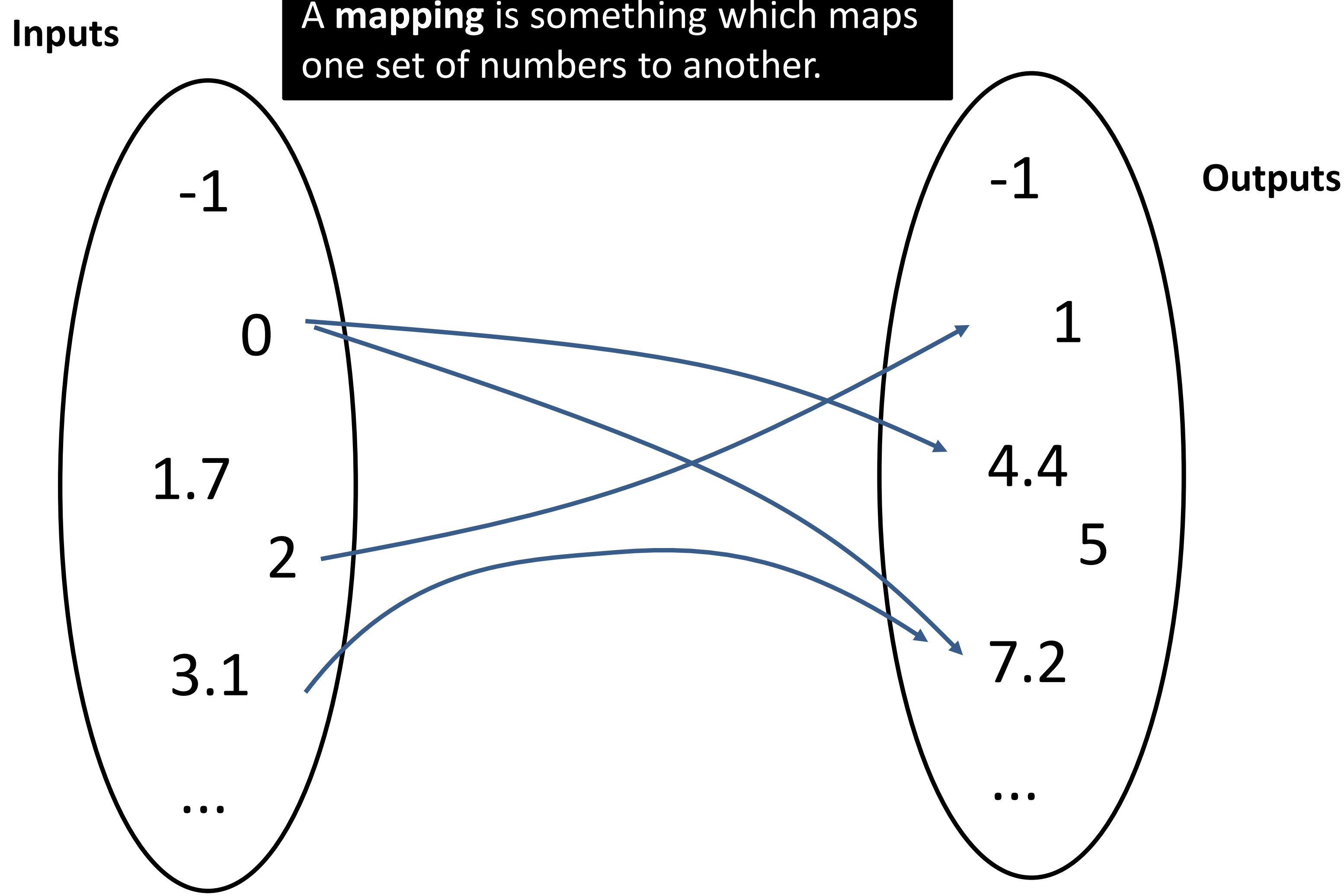
What is a mapping?

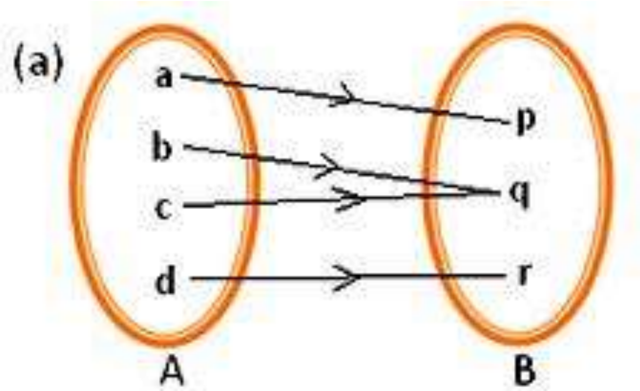
A **mapping** is something which maps one set of numbers to another.



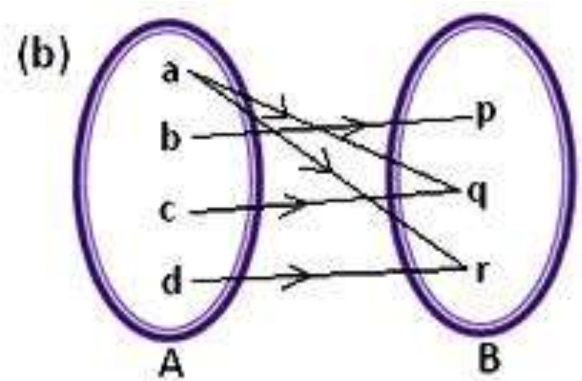
The **domain** is the set of possible inputs.

The **range** is the set of possible outputs.

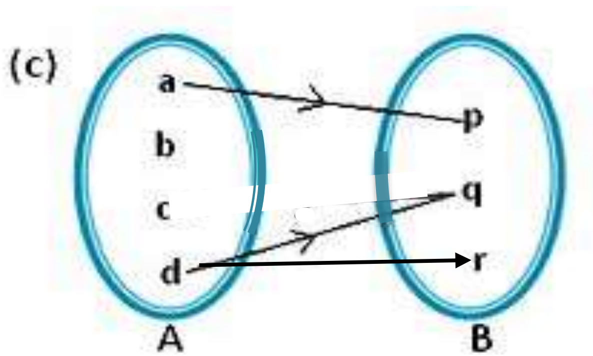
Types of mapping:



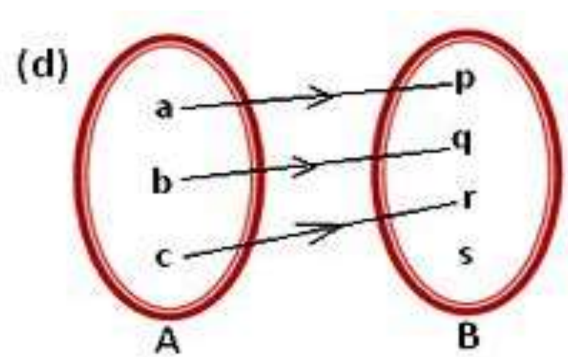
Many to one



Many to many



One to many



One to one

What is a function?

✎ A function is a mapping such that every element of the domain is mapped to exactly one element of the range. Any mapping that is one to one or many to one is also a function. Every x value only produces one y value, although a y value could have come from more than one x value.

Can you think of any examples??

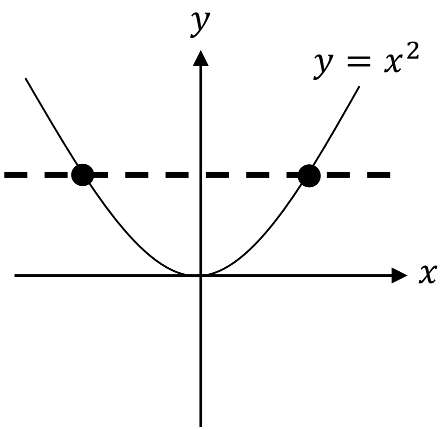
Notation: $f: x \to 2x + 1$ $f(x) = 2x + 1$

$f(x)$ refers to the **output** of the function.

One-to-one vs Many-to-one

While functions permit an input only to be mapped to one output, there's nothing stopping multiple different inputs mapping to the same output.

Type	Description	Example
Many-to-one function	?	?
One-to-one function	?	?

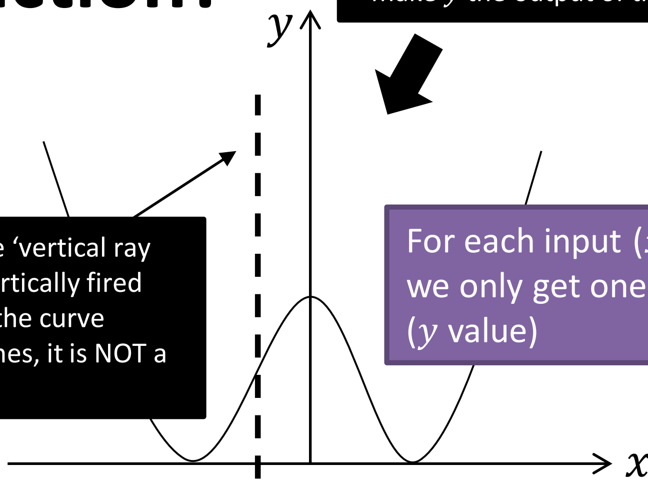


You can use the 'horizontal ray test' to see if a function is one-to-one or many-to-one.

What is a function?

Function?

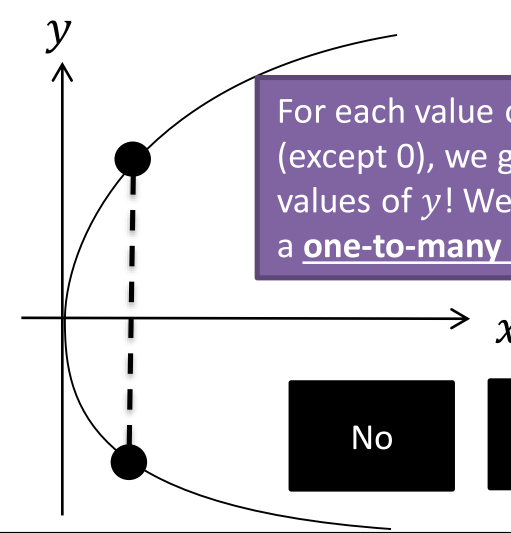
Note: We can illustrate a mapping/function graphically, by plotting a point (x, y) if x maps to y. For this reason we write $y = f(x)$ to mean "make y the output of the function".



Tip: Use the 'vertical ray test'. If a vertically fired ray can hit the curve multiple times, it is NOT a function.

For each input (x value), we only get one output (y value)

No | Yes



For each value of x (except 0), we get two values of y! We say this is a **one-to-many mapping**.

No | Yes

$$f(x) = \sqrt{x}$$

Domain: $x \in \mathbb{R}$

No | Yes

We can't square root a negative number, but the input set is $\mathbb{R}$, so some inputs don't map to a value.

$$f(x) = 2^x$$

Domain: $x \in \mathbb{R}$
(i.e. all real values)

No | Yes

$$f(x) = \pm\sqrt{x}$$

Domain: $x \geq 0$

No | Yes

$f(4) = 2$ but $f(4) = -2$ also. This is one-to-many so not a function.

Domain/Range

It is important that you can identify the range for **<u>common graphs</u>, using a suitable sketch**:

$f(x) = x^2, \qquad x \in \mathbb{R}$

Range: ?

$f(x) = \frac{1}{x}, \qquad x \in \mathbb{R}, x \neq 0$

Range: ?

$f(x) = \ln x, \qquad x \in \mathbb{R}, x > 0$

Range: ?

$f(x) = e^x, \qquad x \in \mathbb{R},$

Range: ?

$f(x) = x^2 + 2x + 9, \qquad x \in \mathbb{R}$

Range: ?

Be careful in noting the domain – it may be 'restricted', which similarly restricts the range. Again, use a sketch!

$f(x) = x^2, \qquad x \in \mathbb{R}, -1 \leq x \leq 4$

Range: ?

Further Examples

It is often helpful to sketch the function to reason about the range.

Find the range of each of the following functions.

a) $f(x) = 3x - 2$, domain $\{1,2,3,4\}$

b) $g(x) = x^2$, domain $\{x \in \mathbb{R}, -5 \leq x \leq 5\}$

c) $h(x) = \frac{1}{x}$, domain $\{x \in \mathbb{R}, 0 < x \leq 3\}$

State if the functions are one-to-one or many-to-one.

We use x to refer to the input, and $f(x)$ to refer to the output.
Thus your ranges should be in terms of $f(x)$.

a

?

b

?

c

?

Piecewise Functions

A 'piecewise function' is one which is defined in parts: we can use different rules for different intervals within the domain.

The function $f(x)$ is defined by

$$f: x \to \begin{cases} 5 - 2x, & x < 1 \\ x^2 + 3, & x \geq 1 \end{cases}$$

a) Sketch $y = f(x)$, and state the range of $f(x)$.
b) Solve $f(x) = 19$

a

?

b

?

Piecewise Functions

Example 2

The function s is defined by

$$s(x) = \begin{cases} x^2 - 6, & x < 0 \\ 10 - x, & x \geqslant 0 \end{cases}$$

a Sketch $y = s(x)$.

b Find the value(s) of a such that $s(a) = 43$.

c Solve $s(x) = x$.

?

Test Your Understanding

Edexcel C4 June 2012 Q6a

The function f is defined by

$$f: x \to e^x + 2, \qquad x \in \mathbb{R}$$

State the range of f.

?

Edexcel C4 June 2010 Q4d

The function g is defined by

$$g: x \to x^2 - 4x + 1, \qquad x \in \mathbb{R}, 0 \leq x \leq 5$$

Find the range of g.

Hint: Identify the minimum point first, as this may or may not affect the range.
Extra Hint: Carefully consider the stated domain.

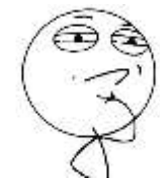
?

Exercise 2B

Pearson Pure Mathematics Year 2/AS
Pages 30-32

Just for your interest…

What is the difference between the notation $f(x)=2x+1$ and $f: x \to 2x+1$?



$f: x \to 2x+1$ means "the value of f is a mapping from x to $2x+1$".

You're used to variables, e.g. x, representing numerical values. But we've also seen that the value of a variable can be a vector, e.g. $\boldsymbol{a}=\binom{2}{1}$, sets, e.g. $A=\{1,2,3\}$ and so on. So when we use f on its own, its 'value' is a mapping, in this case with the value $x \to 2x+1$.
This notation therefore places more emphasis on the <u>value</u> of f, and its 'value' <u>as a mapping</u>.

Consequence 1

To solve an equation means to find the values of the variables, e.g. the "solution" of $2x+1=5$ is $x=2$.

To solve a functional equation means to find the 'values' of f.

Solve $f(x+y)=f(x)f(y)$

One solution to this equation is $f: x \to 2^x$ because $f(x+y)=2^{x+y}$ and $f(x)f(y)=2^x 2^y=2^{x+y}$.
To fully solve this functional equation means to find **all** functions which satisfy the equation.
See http://www.drfrostmaths.com/resources/resource.php?rid=165

$f(x)=2x+1$ means "the output of f is $2x+1$".

It's easy to think that the notation "$f(x)$" refers to the function. It doesn't! The f is the function, and the "(x)" appendage obtains the <u>output</u> of the function when the input is x. Therefore $f(x)$ refers specifically to the output of the function, which is why we write the range of a function in terms of $f(x)$ (and not in terms of f).
This notation therefore places more emphasis on the <u>output</u> of f.

Consequence 2

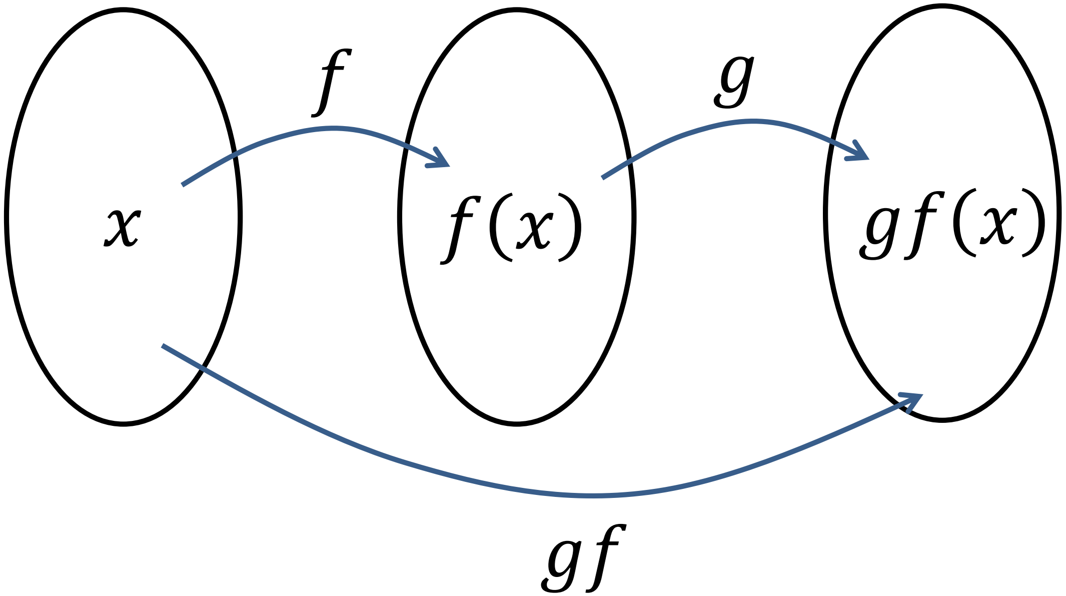
A bit of Computer Science!
In many programming languages, we can pass functions as the parameters of a method, when a variable is allowed to have a function as its value.

We could code a function `map` which takes a list, say a, and applies a function f to each item of this list.
e.g. `map(x→x+1, [1,2,3])` would output `[2,3,4]`.

```
function map(f, a) {
    let b be a new list
    for(i from 1 to size(a)) {
        b_i = f(a_i)
    }
    return b
}
```

Composite Functions

Sometimes we may apply multiple functions in succession to an input. These combined functions are known as a **composite function**.



✎ $gf(x)$ means $g(f(x))$, i.e. f is applied first, then g.

Examples

Let $f(x) = x^2 + 1$, and $g(x) = 4x - 2$.
What is…

$fg(2)$? $=$?

$fg(x)$? ?

$gf(x)$? ?

$f^2(x)$? ?

$f^2(x)$ means $ff(x)$

Solve $gf(x) = 38$?

Further Examples

The functions f and g are defined by

$$f: x \to |2x - 8|$$

$$g: x \to \frac{x+1}{2}$$

a) Find $fg(3)$ b) Solve $fg(x) = x$

a

?

b

?

Test Your Understanding

Edexcel C4 June 2013(R) Q4

The functions and f and g are defined by

$$f: x \to 2|x| + 3, \qquad x \in \mathbb{R}$$
$$g: x \to 3 - 4x, \qquad x \in \mathbb{R}$$

b) Find $fg(1)$

d) Solve the equation

$$gg(x) + [g(x)]^2 = 0$$

Edexcel C4 June 2012 Q6

The functions f and g are defined by

$$f: x \to e^x + 2, \qquad x \in \mathbb{R}$$
$$g: x \to \ln x, \qquad x > 0$$

b) Find $fg(x)$, giving your answer in its simplest form.

? b

? d

?

Exercise 2C

Pearson Pure Mathematics Year 2/AS
Pages 34-35

Extension

1 [MAT 2014 1F]
The functions S and T are defined for real numbers by $S(x) = x + 1$ and $T(x) = -x$.
The function S is applied s times and the function T is applied t times, in some order, to produce the function

$$F(x) = 8 - x$$

It is possible to deduce that:

i) $s = 8$ and $t = 1$
ii) s is odd and t is even.
iii) s is even and t is odd.
iv) s and t are powers of 2.
v) none of the above.

?

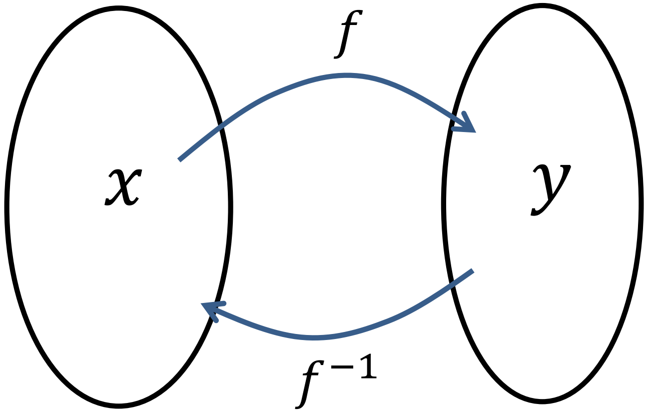
2 [MAT 2012 Q2]
Let $f(x) = x + 1$ and $g(x) = 2x$.
i) Show that $f^2g(x) = gf(x)$
ii) Note that $gf^2g(x) = 4x + 4$
Find all the other ways of combining f and g that result in the function $4x + 4$.
iii) Let $i, j, k \geq 0$ be integers. Determine the function

$$f^i g f^j g f^k(x)$$

iv) Let $m \geq 0$ be an integer. How many different ways of combining the functions f and g are there that result in the function $4x + 4m$?

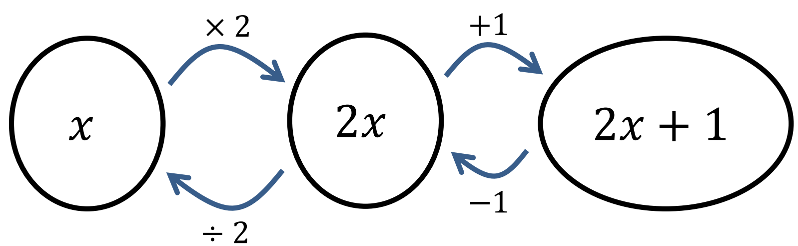
?

Inverse Functions



An inverse function f^{-1} **does the opposite of the original function**. For example, if $f(4) = 2$, then $f^{-1}(2) = 4$.

If $f(x) = 2x + 1$, we could do the opposite operations within the function in reverse order to get back to the original input:



Thus $f^{-1}(x) = \frac{x-1}{2}$

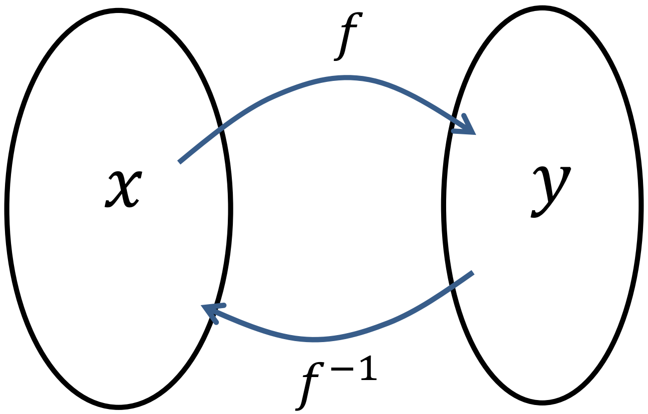
Notation: Just like f^2 means "apply f twice", f^{-1} means "apply f -1 times", i.e. once backwards!
This is why we write $\sin^{-1}(x)$ to mean "inverse sin".

This has appeared in exams before.

Explain why the function must be one-to-one for an inverse function to exist:

?

More on Inverse Functions



In the original function, we have the **output y in terms of the input x**, e.g. $y = 2x + 1$

Therefore if we **change the subject to get x in terms of y**, then we have the input in terms of the output, i.e. the inverse function!

$$x = \frac{y-1}{2}$$

However, we tend to write a function in terms of x, so would write;

$$f^{-1}(x) = \frac{x-1}{2}$$

If $f(x) = 3 - 4x$, find $f^{-1}(x)$

?

If $f(x) = \frac{x+2}{2x-1}$, $x \neq \frac{1}{2}$, determine $f^{-1}(x)$

?

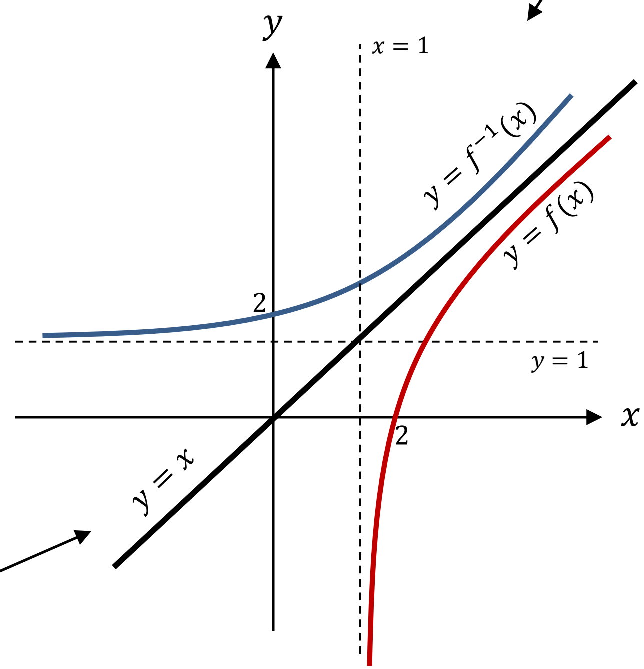
Graphing an Inverse Function

We saw that the inverse function effectively swaps the input x and output y. Thus the **x and y axis are swapped** when sketching the original function and its inverse.

And since the set of inputs and set of outputs is swapped…

✎ The domain of $f(x)$ is the range of $f^{-1}(x)$ and vice versa.

Notice that x-intercepts become y-intercepts, and vertical asymptotes become horizontal ones.



$y = f(x)$ and $y = f^{-1}(x)$ have the line $y = x$ as a line of symmetry.

Domain of f:

?

Range of f^{-1}:

?

The domain of the function is the same as the range of the inverse, but remember that we write a domain in terms of x, but a range in terms of $f(x)$ or $f^{-1}(x)$.

Example

If $g(x)$ is defined as $g(x) = \sqrt{x-2}\ \{x \in \mathbb{R}, x \geq 2\}$

a) Find the range of $g(x)$.

b) Calculate $g^{-1}(x)$

c) Sketch the graphs of both functions.

d) State the domain and range of $g^{-1}(x)$.

a ?

b ?

c ?

d ?

Further Example

The function is defined by $f(x) = x^2 - 3,\ x \in \mathbb{R},\ x \geq 0$.

a) Find $f^{-1}(x)$

b) Sketch $y = f(x)$ and $y = f^{-1}(x)$ and state the domain of f^{-1}.

c) Solve the equation $f(x) = f^{-1}(x)$.

a

?

b

?

c

?

Test Your Understanding

Edexcel C4 June 2012 Q6

The function f is defined by

$$f: x \to e^x + 2, \quad x \in \mathbb{R}$$

(d) Find f^{-1}, the inverse function of f, stating its domain.

(e) On the same axe sketch the curves with equation $y = f(x)$ and $y = f^{-1}(x)$, giving the coordinates of all the points where the curves cross the axes.

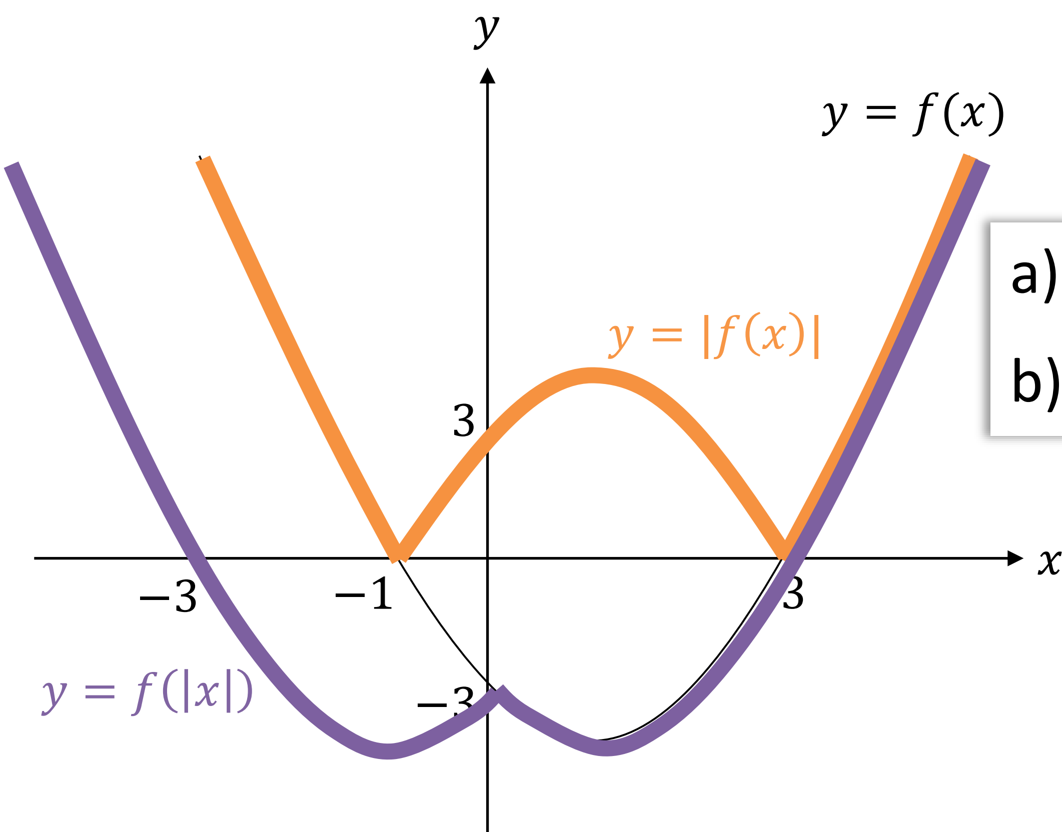
(d) ?

(e) ?

Exercise 2D

Pearson Pure Mathematics Year 2/AS
Pages 38-39

Sketching $y = |f(x)|$ and $y = f(|x|)$



This is a sketch of $y = f(x)$ where $f(x) = (x - 3)(x + 1)$

a) Sketch $y = |f(x)|$ Sketch >

b) Sketch $y = f(|x|)$ Sketch >

The | … | is outside the function so affects the y value. Any negative y values will be made positive, so any parts of the graph below the x-axis are flipped upwards.
Ensure the y-intercept is indicated.

When $x = -3$ for example, this becomes +3 before being fed into the function, therefore we actually use the y value when x would have been 3 instead of the original -3.
The result is that the graph left of the y-axis is discarded and the graph right of it copied over by reflection in the y-axis.

Test Your Understanding

Edexcel C4 June 2012 Q4

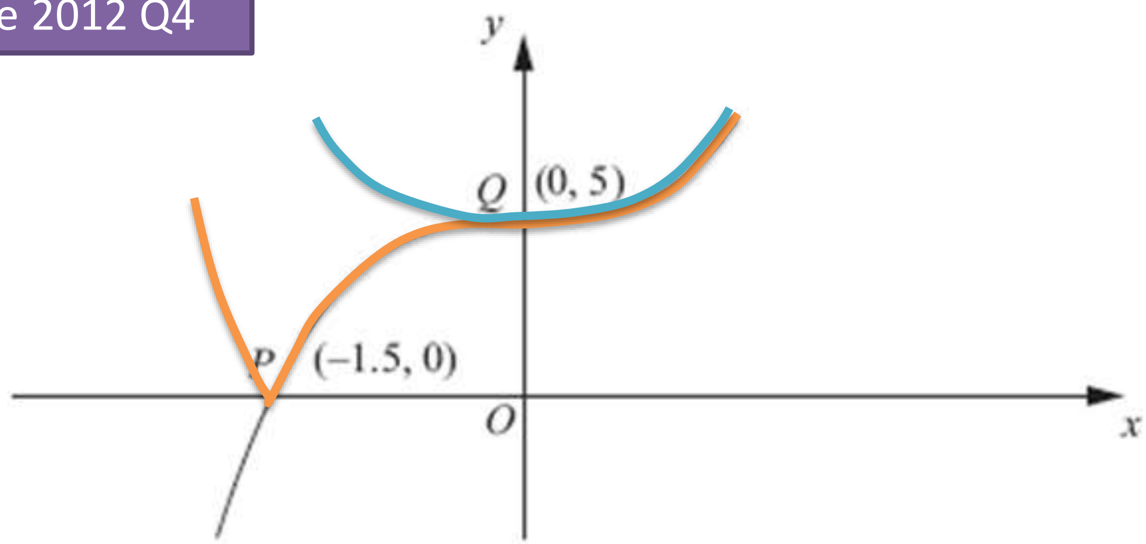


Figure 2 shows part of the curve with equation $y = f(x)$.
The curve passes through the points $P(-1.5, 0)$ and $Q(0, 5)$ as shown.

On separate diagrams, sketch the curve with equation

(*a*) $y = |f(x)|$ Sketch >

(2)

(*b*) $y = f(|x|)$ Sketch >

(2)

Indicate clearly on each sketch the coordinates of the points at which the curve crosses or meets the axes.

Further Test Your Understanding

[Textbook] Sketch for $-2\pi \leq x \leq 2\pi$:

a) $y = |\sin(x)|$

b) $y = \sin(|x|)$

a

?

b

?

Exercise 2E

Pearson Pure Mathematics Year 2/AS
Pages 42-44

Extension

1 [SMC 2008 Q25] What is the area of the polygon forms by all the points (x, y) in the plane satisfying the inequality $\big||x|-2\big|+\big||y|-2\big| \leq 4$?

A 24 B 32 C 64 D 96 E 112

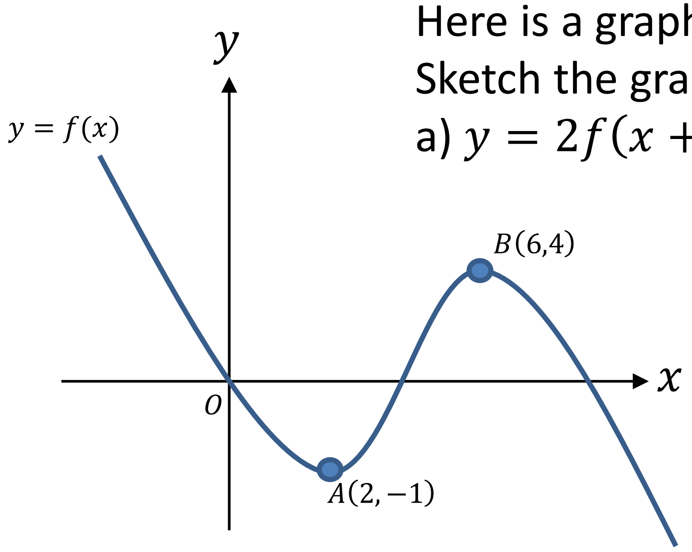
?

Combining Transformations

RECAP:

	Affects which axis?	What we expect or opposite?
Change **inside** $f(\,)$	?	?
Change **outside** $f(\,)$	?	?

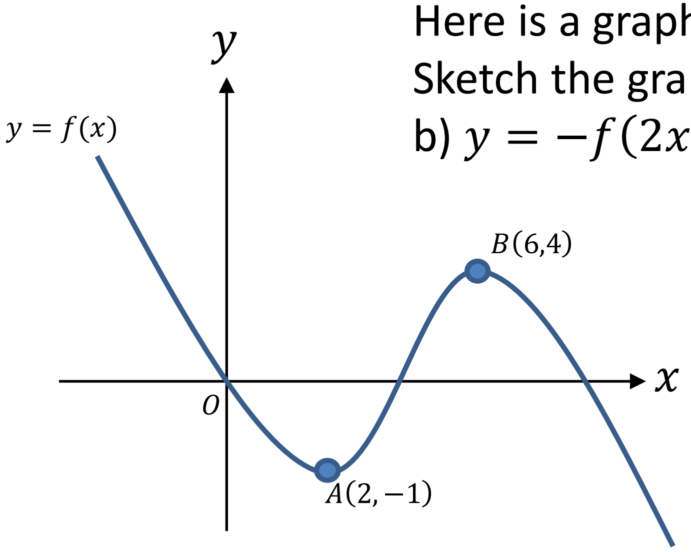
There is nothing new here relative to Year 1, except that you might have to do more than one transformation…



Here is a graph of $y = f(x)$.
Sketch the graph of:
a) $y = 2f(x+2)$

?

Combining Transformations



Here is a graph of $y = f(x)$.
Sketch the graph of:
b) $y = -f(2x)$

? b

c) $y = |f(-x)|$

? c

Test Your Understanding

C4 June 2011 Q3

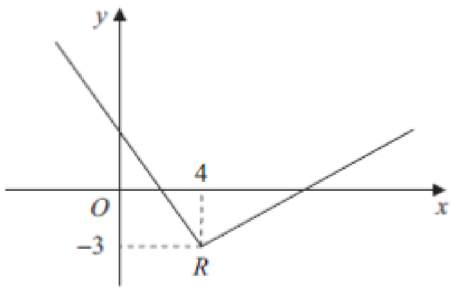


Figure 1

Figure 1 shows part of the graph of $y = \mathrm{f}(x)$, $x \in \mathbb{R}$.

The graph consists of two line segments that meet at the point R $(4, -3)$, as shown in Figure 1.

Sketch, on separate diagrams, the graphs of

(*a*) $y = 2\mathrm{f}(x + 4)$, **(3)**
(*b*) $y = |\mathrm{f}(-x)|$. **(3)**
On each diagram, show the coordinates of the point corresponding to R.

(a)

? a

(b)

? b

What if two x changes or two y changes?

$$y = 2f(x) + 1$$

?

$$y = f(2x + 1)$$

You will not get multiple x transformations in your exam, but theoretically…

?

Sketch $y = \ln(1 - 2x)$

?

Exercise 2F

Pearson Pure Mathematics Year 2/AS
Pages 47-48

Solving Modulus Problems

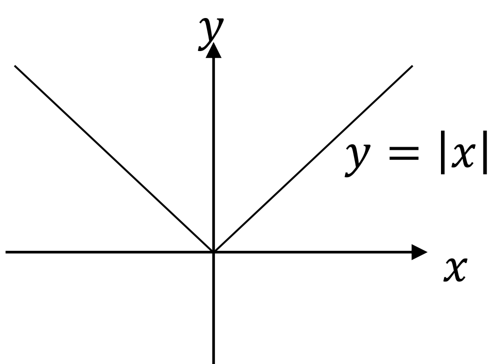
[Textbook] Given the function $f(x) = 3|x-1| - 2, x \in \mathbb{R}$,

(a) Sketch the graph of $y = f(x)$

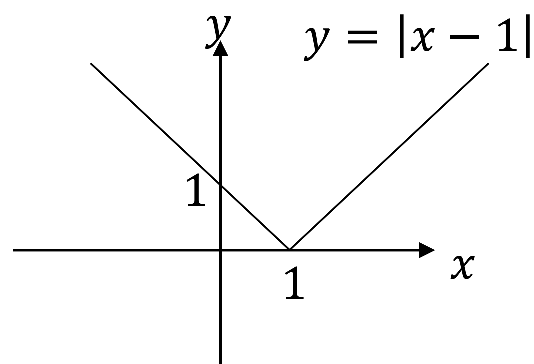
(b) State the range of f.

(c) Solve the equation $f(x) = \frac{1}{2}x + 3$

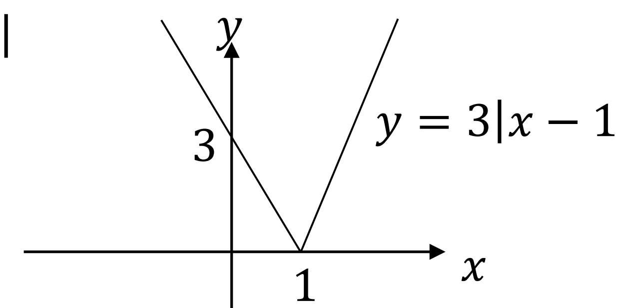
a It is often helpful to **sketch the graph in stages** as we apply more transformations:



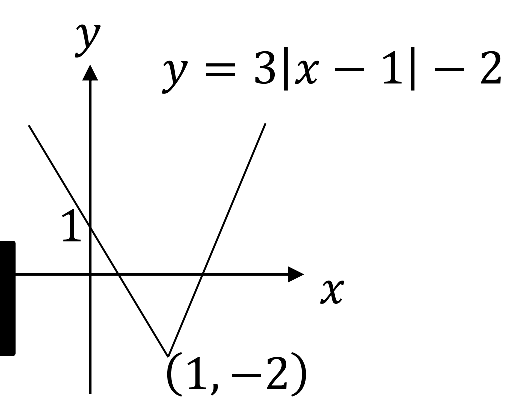
Start with the 'simplest' version of the graph, $y = |x|$



-1 is 'inside' function so translate 1 right.



3 is outside modulus function so affects y values.

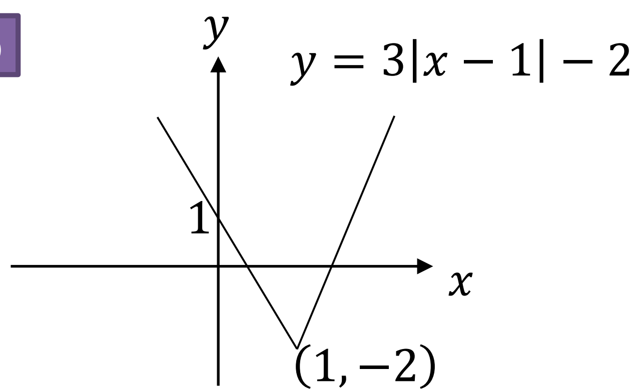


-2 is outside modulus function so translate 2 down.

Solving Modulus Problems

[Textbook] Given the function $f(x) = 3|x - 1| - 2, x \in \mathbb{R}$,
(a) Sketch the graph of $y = f(x)$
(b) State the range of f.
(c) Solve the equation $f(x) = \frac{1}{2}x + 3$

b



?

Solving Modulus Problems

[Textbook] Given the function $f(x) = 3|x-1| - 2, x \in \mathbb{R}$,

(a) Sketch the graph of $y = f(x)$

(b) State the range of f.

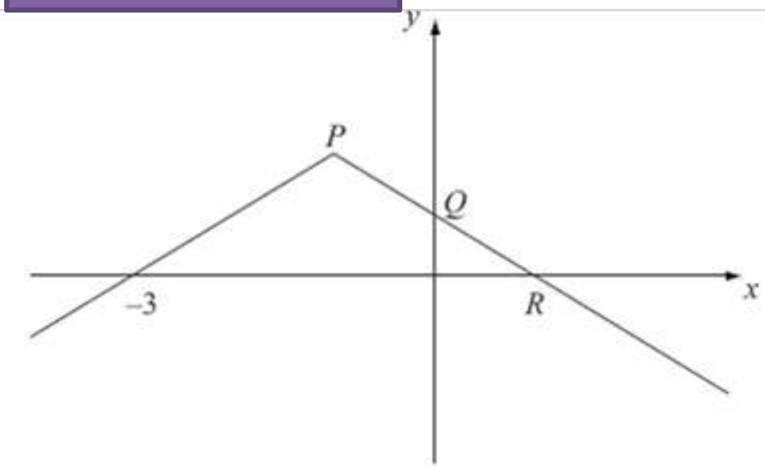
(c) Solve the equation $f(x) = \frac{1}{2}x + 3$

c

?

Test Your Understanding

C4 June 2008 Q3



You can sketch this function by starting with $y = |x|$ and gradually transform it as per the previous example.

Given that $f(x) = 2 - |x + 1|$,

(*c*) find the coordinates of the points P, Q and R, **(3)**

(*d*) solve $f(x) = \frac{1}{2}x$. **(5)**

a

?

b

?

Exercise 2G

Pearson Pure Mathematics Year 2/AS
Pages 51-52

Extension

[MAT 2006 1I]
The equation $|x| + |x - 1| = 0$ has how many solutions?

?