P2 Chapter 11 :: Integration

jfrost@tiffin.kingston.sch.uk

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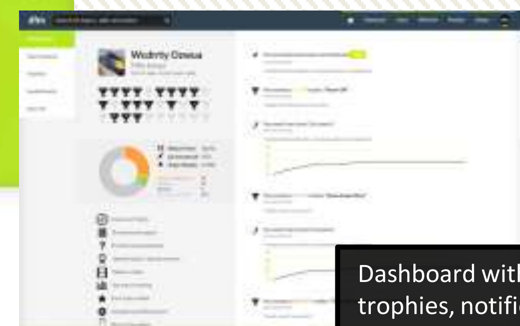

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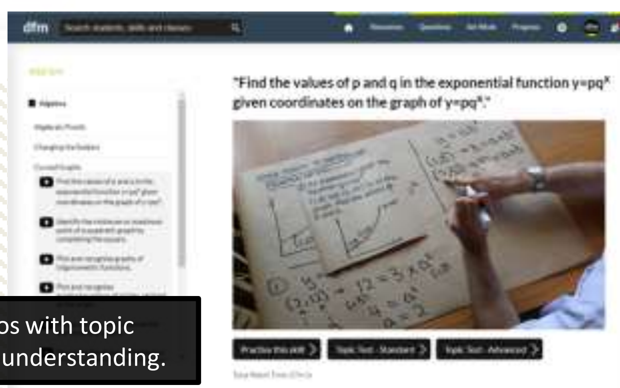

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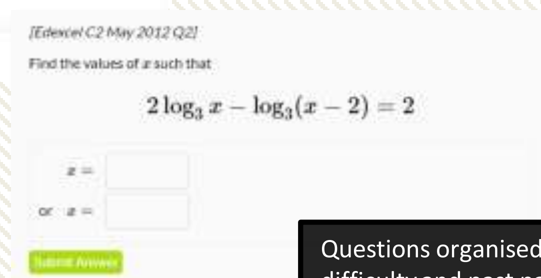


Questions organised by topic, difficulty and past paper.

Overview

In this chapter, you'll be able to integrate a significantly greater variety of expressions, and be able to solve differential equations.

Integration by standard result
(There's certain expressions you're expected to know straight off.)

$$\int \sec^2 x \, dx = \tan x + C$$

Integration by 'reverse chain rule'
(We imagine what would have differentiated to get the expression.)

$$\int \cos 4x \;\; dx = \frac{1}{4} \sin 4x + C$$

$$\int \sin^3 x \cos x \, dx = \frac{1}{4} \sin^4 x + C$$

Integration by substitution
(We make a substitution to hopefully make the expression easier to integrate)

$$\int x\sqrt{2x+5}\, dx$$

Let $u = 2x + 5 \quad \rightarrow \quad x = \frac{u-5}{2}$

$\frac{du}{dx} = 2 \quad \rightarrow \quad dx = \frac{1}{2} u$

$$\int x\sqrt{2x+5}\, dx = \int \frac{u-5}{2} \frac{1}{2} du = \cdots$$

Integration by parts
(Allows us to integrate a product, just as the product rule allowed us to differentiate one)

$$\int x \cos x \, dx$$

$u = x \quad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 1 \quad v = \sin x$

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x$$

Overview

Integrating partial fractions
(We split into partial fractions first so each fraction easier to integrate)

$$\int \frac{3x+5}{(x+1)(x+2)}\,dx$$

Approximating areas using the trapezium rule
(Instead of integrating, we split the area under the graph into trapeziums and use these to approximate the area)

Solving Differential Equations
(Solving here means to find one variable in terms of another without derivatives present)

$$\frac{dV}{dt} = -kV$$
$$\int \frac{1}{V}\,dV = \int -k\;dt$$
$$\ln V = -kt + C$$
$$V = e^{-kt+C} = Ae^{-kt}$$

Notes for teachers: The chapter is the same as the old C4 integration except:

- Volumes of Revolution has been removed and is now (slightly oddly) a chapter in itself in Further Maths.
- Integration using parametric equations IS still in the syllabus (but was removed from earlier versions of the new Pearson textbooks)

SKILL #1: Integrating Standard Functions

There's certain results you should be able to integrate straight off, by just thinking about the opposite of differentiation.

From the Formula Booklet

First Principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x)$	$f'(x)$
$\tan kx$	$k \sec^2 kx$
$\sec kx$	$k \sec kx \tan kx$
$\cot kx$	$-k \operatorname{cosec}^2 kx$
$\operatorname{cosec} kx$	$-k \operatorname{cosec} kx \cot kx$

y	$\int y \, dx$
x^n	?
e^x	?
$\frac{1}{x}$	?
$\cos x$	?
$\sin x$	?
$\sec^2 x$	?
$cosec\ x \cot x$	?
$cosec^2 x$	?
$\sec x \tan x$	?

The $|x|$ has to do with problems when x is negative (when $\ln x$ is not defined)

Remember my memorisation trick of picturing sin above cos from C3, so that 'going down' is differentiating and 'going up' is integrating, and we change the sign if the wrong way round.

It's vital you remember this one.

Have a good stare at this slide before turning your paper over – let's see how many you remember…

Quickfire Questions (without cheating!)

$$\int \sec x \tan x \; dx = \text{?}$$

$$\int \sin x \; dx = \text{?}$$

$$\int cosec^2\, x \, dx = \text{?}$$

$$\int -\cos x \; dx = \text{?}$$

Quickfire Questions (without cheating!)

$$\int \sec^2 x \ dx = \boxed{?}$$

$$\int cosec\, x \cot x \ dx = \boxed{?}$$

$$\int \frac{1}{x} \ dx = \boxed{?}$$

$$\int -\sin x \ dx = \boxed{?}$$

Quickfire Questions (without cheating!)

$$\int cosec^2\, x\, dx = \boxed{?}$$

$$\int \sin x\ \ dx = \boxed{?}$$

$$\int \sec x \tan x\ \ dx = \boxed{?}$$

$$\int \cos x\ \ dx = \boxed{?}$$

Test Your Understanding

$$\int 2\cos x + \frac{3}{x} - \sqrt{x}\, dx =$$

?

$$\int \frac{\cos x}{\sin^2 x}\, dx =$$

?

Hint: What 'reciprocal' trig functions does this simplify to?

[Textbook] Given that $\int_a^{3a} \frac{2x+1}{x}\, dx = \ln 12$, find the exact value of a.

?

Important Notes:
We can simplify:

$$\frac{x+1}{x} \equiv \frac{x}{x} + \frac{1}{x} \equiv 1 + \frac{1}{x}$$

However it is **NOT** true that:

$$\frac{x}{x+1} \equiv \frac{x}{x} + \frac{x}{1}$$

In my experience students often fail to spot when they can split up a fraction to then integrate.

Exercise 11A

Pearson Pure Mathematics Year 2/AS
Pages 295-296

SKILL #2: Integrating $f(ax+b)$

$$\frac{d}{dx}(\sin(3x+1)) = ?$$

Therefore:

$$\int \cos(3x+1)\ dx = ?$$

For any expression where inner function is $ax+b$, integrate as before and $\div a$.

$$\int f'(ax+b)dx = \frac{1}{a}f(ax+b)+C$$

Quickfire:

$$\int e^{3x}\ dx = ?$$

$$\int \frac{1}{5x+2}dx = ?$$

$$\int 2\sec^2(3x-2)\ dx = ?$$

$$\int (3x+4)^3\ dx = ?$$

$$\int \sin(1-5x)\ dx = ?$$

$$\int \frac{1}{3(4x-2)^2}dx = ?$$

$$\int (10x+11)^{12} = ?$$

Fro Tip: For $\int (ax+b)^n\ dx$, ensure you divide by the $(n+1)$ <u>and</u> the a

Check Your Understanding

$$\int e^{3x+1} dx = \text{?}$$

$$\int \frac{1}{1-2x} dx = \text{?}$$

$$\int (4-3x)^5 dx = \text{?}$$

$$\int \sec(3x)\tan(3x)\, dx = \text{?}$$

Exercise 11B

1 a $\int \sin(2x+1)\,dx =$?

c $\int 4e^{x+5}\,dx =$?

e $\int cosec^2 3x\,dx =$?

f $\int \sec 4x \tan 4x\,dx =$?

g $\int 3\sin\left(\frac{1}{2}x+1\right)dx =$?

h $\int cosec\ 2x \cot 2x\,dx =$?

2 a $\int e^{2x} - \frac{1}{2}\sin(2x-1)\,dx$

$=$?

b $\int (e^x+1)^2\,dx =$?

c $\int \sec^2 2x\,(1+\sin 2x)\,dx =$?

d $\int \frac{3-2\cos\left(\frac{1}{2}x\right)}{\sin^2\left(\frac{1}{2}x\right)}\,dx =$?

e $\int e^{3-x} + \sin(3-x) + \cos(3-x)\,dx$

$=$?

3 a $\int \frac{1}{2x+1}\,dx =$?

b $\int \frac{1}{(2x+1)^2}\,dx =$?

c $\int (2x+1)^2\,dx =$?

d $\int \frac{3}{4x-1}\,dx =$?

f $\int \frac{3}{(1-4x)^2}\,dx =$?

h $\int \frac{3}{(1-2x)^3}\,dx =$?

j $\int \frac{5}{3-2x}\,dx =$?

4 a $\int 3\sin(2x+1) + \frac{4}{2x+1}\,dx$

$=$?

c $\int \frac{1}{\sin^2 2x} + \frac{1}{1+2x} + \frac{1}{(1+2x)^2}\,dx$

$=$?

SKILL #3: Integrating using Trig Identities

Some expressions, such as $\sin^2 x$ and $\sin x \cos x$ can't be integrated directly, but we can use one of our trig identities to replace it with an expression we can easily integrate.

Q Find $\int \sin^2 x \; dx$

?

Q Find $\int \sin 3x \cos 3x \; dx$

?

Q Find $\int \cos^2 x \; dx$

?

Q Find $\int \tan^2 x \; dx$

?

Check Your Understanding

Q Find $\int (\sec x + \tan x)^2 \; dx$

?

Q Find $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 3x \;\; dx$

?

Exercise 11C

Pearson Pure Mathematics Year 2/AS
Page 300

SKILL #4: Reverse Chain Rule

There's certain more complicated expressions which look like the result of having applied the chain rule. I call this process '**<u>consider then scale</u>**':

1. **<u>Consider</u> some expression that will differentiate to something similar to it.**
2. **Differentiate, and adjust for any <u>scale</u> difference.**

$$\int x(x^2+5)^3\,dx$$

The first x looks like it arose from differentiating the x^2 inside the brackets.

?

$$\int \cos x \sin^2 x\ \,dx$$

The $\cos x$ probably arose from differentiating the sin.

?

$$\int \frac{2x}{x^2+1}\,dx$$

The $2x$ probably arose from differentiating the x^2.

?

SKILL #4: Reverse Chain Rule

✐ **Integration by Inspection/Reverse Chain Rule**: Use common sense to **consider some expression** that would differentiate to the expression given. Then **scale** appropriately.

Common patterns:

$$\int k \frac{f'(x)}{f(x)} dx \rightarrow \quad Try \ \ln|f(x)|$$

In words: "If the bottom of a fraction differentiates to give the top (forgetting scaling), try ln of the bottom".

$$\int k f'(x)[f(x)]^n \rightarrow \quad Try \ [f(x)]^{n+1}$$

$$\int \frac{x^2}{x^3+1} \, dx$$

?

$$\int x \, e^{x^2+1} \, dx$$

?

Quickfire

In your head!

$$\int \frac{4x^3}{x^4 - 1}\, dx = \text{?}$$

$$\int \frac{\cos x}{\sin x + 2}\, dx = \text{?}$$

$$\int \cos x \; e^{\sin x}\, dx = \text{?}$$

$$\int \cos x\, (\sin x - 5)^7\, dx = \text{?}$$

$$\int x^2 (x^3 + 5)^7 = \text{?}$$

Not in your head…

$$\int \frac{x}{(x^2 + 5)^3}\, dx = \text{?}$$

Fro Tip: If there's as power around the whole denominator, DON'T use ln: reexpress the expression as a product. e.g. $x(x^2 + 5)^{-3}$

$\sin^n x \cos x$ vs $\sec^n x \tan x$

Notice when we differentiate $\sin^5 x$, then power decreases:

$$\frac{d}{dx}\left(\sin^5 x\right) = \boxed{?}$$

However, when we differentiate $\sec^5 x$:

$$\frac{d}{dx}\left((\sec x)^5\right) = \boxed{?}$$

Notice that the power of sec didn't go down. Keep this in mind when integrating.

$$\int \sec^4 x \tan x \; dx$$

?

Test Your Understanding

$$\int \sin x\,(\cos x + 1)^5\; dx$$

?

$$\int \frac{cosec^2 x}{(2+\cot x)^3}\,dx$$

?

$$\int \frac{\sec^2 2x}{\tan 2x + 1}\; dx$$

?

$$\int x(x^2+2)^3\; dx$$

?

$$\int 5\tan x\sec^2 x\;\; dx$$

?

Exercise 11D

Pearson Pure Mathematics Year 2/AS
Page 302-303

Extension:

[STEP I 2013 Q4]
(i) Show that, for $n > 0$,
$\int_0^{\frac{1}{4}\pi} \tan^n x \sec^2 x \ dx = \frac{1}{n+1}$ and $\int_0^{\frac{1}{4}\pi} \sec^n x \tan x \ dx = \frac{\left(\sqrt{2}\right)^n - 1}{n}$
(ii) Evaluate the following integrals:
$\int_0^{\frac{1}{4}\pi} x \sec^4 x \tan x \ dx$ and $\int_0^{\frac{1}{4}\pi} x^2 \sec^2 x \tan x \ dx$

Solutions: (ii) ?

SKILL #5: Integration by Substitution

For some integrations involving a complicated expression, we can make a substitution to turn it into an equivalent integration that is simpler. We wouldn't be able to use 'reverse chain rule' on the following:

Q Use the substitution $u = 2x + 5$ to find $\int x\sqrt{2x+5}\ dx$

The aim is to completely remove any reference to x, and replace it with u. We'll have to work out x and dx so that we can replace them.

STEP 1: Using substitution, work out x and dx (or variant)	?
STEP 2: Substitute these into expression.	?
STEP 3: Integrate simplified expression.	?
STEP 4: Write answer in terms of x.	?

How can we tell what substitution to use?

In Edexcel you will usually be given the substitution!
However in some other exam boards, and in STEP, you often aren't.
There's no hard and fast rule, but it's often helpful to replace to replace expressions inside roots, powers or the denominator of a fraction.

Sensible substitution:

$$\int \cos x \sqrt{1+\sin x}\, dx$$ $\boldsymbol{u} =$?

$$\int 6x\, e^{x^2} dx$$ $\boldsymbol{u} =$?

$$\int \frac{xe^x}{1+x} dx$$ $\boldsymbol{u} =$?

$$\int e^{\frac{1-x}{1+x}} dx$$ $\boldsymbol{u} =$?

Another Example

Q Use the substitution $u = \sin x + 1$ to find $\int \cos x \sin x \,(1 + \sin x)^3 \; dx$

STEP 1: Using substitution, work out x and dx (or variant)	?
STEP 2: Substitute these into expression.	?
STEP 3: Integrate simplified expression.	?
STEP 4: Write answer in terms of x.	?

Using substitutions involving implicit differentiation

When a root is involved, it makes thing much tidier if we use $u^2 = \cdots$

Q Use the substitution $u^2 = 2x + 5$ to find $\int x\sqrt{2x+5}\ dx$

STEP 1: Using substitution, work out x and dx (or variant)	?
STEP 2: Substitute these into expression.	?
STEP 3: Integrate simplified expression.	?
STEP 4: Write answer in terms of x.	?

This was marginally less tedious than when we used $u = 2x + 5$, as we didn't have fractional powers to deal with.

Test Your Understanding

Edexcel C4 Jan 2012 Q6c

(*c*) Using the substitution $u = 1 + \cos x$, or otherwise, show that

$$\int \frac{2\sin 2x}{(1+\cos x)}\,dx = 4\ln(1+\cos x) - 4\cos x + k,$$

where k is a constant. **(5)**

Hint: You might want to use your double angle formula first.

?

Definite Integration

Now consider:

Q Calculate $\int_0^{\frac{\pi}{2}} \cos x \sqrt{1+\sin x}\ dx$

Use substitution: ?

?

Now because we've changed from x to u, we have to work out what values of u would have given those limits for x:

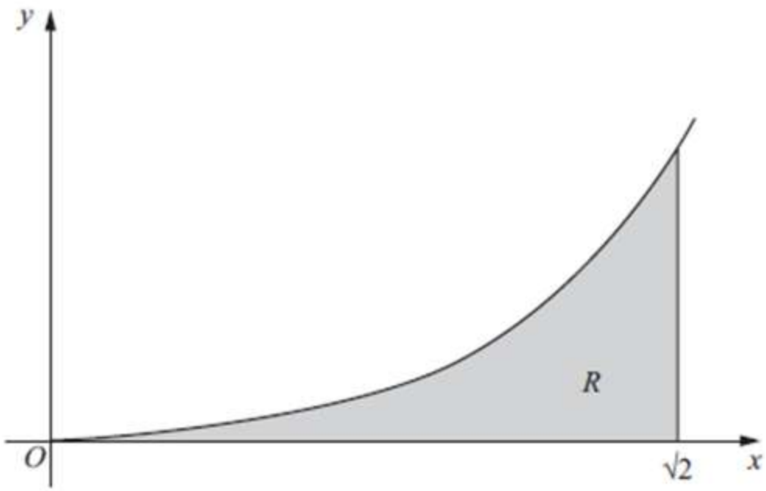
When $x = \frac{\pi}{2}$, $u =$?
When $x = 0$, $u =$

$$\int_0^{\frac{\pi}{2}} \cos x \sqrt{1+\sin x}\ dx =$$?

$=$

Test Your Understanding

Edexcel C4 June 2011 Q4



?

Figure 2 shows a sketch of the curve with equation $y = x^3 \ln(x^2 + 2)$, $x \geq 0$.

The finite region R, shown shaded in Figure 2, is bounded by the curve, the x-axis and the line $x = \sqrt{2}$.

(*c*) Use the substitution $u = x^2 + 2$ to show that the area of R is

$$\frac{1}{2}\int_2^4 (u-2)\ln u \; du.$$

Exercise 11E

Pearson Pure Mathematics Year 2/AS
Page 306-307

SKILL #6: Integration by Parts

$$\int x \cos x \ dx = ?$$

Just as the Product Rule was used to **differentiate the product** of two expressions, we can often use ‘Integration by Parts’ to **integrate a product.**

To integrate by parts:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} \ dx$$

Proof ?

(not needed for exam)

SKILL #6: Integration by Parts

$$\int x \cos x \;\; dx = ?$$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} \, dx$$

?

STEP 1: Decide which thing will be u (and which $\frac{dv}{dx}$).

You're about to differentiate u and integrate $\frac{dv}{dx}$, so the idea is to pick them so differentiating u makes it 'simpler', and $\frac{dv}{dx}$ can be integrated easily.

u will always be the x^n term **UNLESS** one term is $\ln x$.

?

STEP 2: Find $\frac{du}{dx}$ and v.

?

STEP 3: Use the formula.

I just remember it as "uv minus the integral of the two new things timesed together"

Another Example

Q Find $\int x^2 \ln x \ dx$

?

Integrating $\ln x$ and definite integration

Q Find $\int \ln x \; dx$, leaving your answer in terms of natural logarithms.

?

Q Find $\int_1^2 \ln x \, dx$, leaving your answer in terms of natural logarithms.

?

IBP twice! 😐

Q Find $\int x^2 e^x \, dx$

?

Test Your Understanding

Q Find $\int x^2 \sin x \ dx$

?

Test Your Understanding

Q Find $\int_0^{\frac{\pi}{2}} x \sin x \ dx$

?

One final unusual one…

$$\int e^x \sin x \ dx$$

It doesn't actually matter what you make the u and what the $\frac{dv}{dx}$ this time. But the hard part is realising how to 'close the loop' at the end…
Exam Note: This came up in an exam once and caught an awful lot of students (and teachers!) by surprise.

?

Exercise 11F

Pearson Pure Mathematics Year 2/AS
Page 306-307

Extension

$$\int \arcsin(x)\, dx$$

$= \boxed{?}$

(hint: $\frac{dv}{dx} = 1$)

[STEP I 2014 Q2]

(i) Show that $\int \ln(2-x)\,dx = -(2-x)\ln(2-x) + (2-x) + c$, where $x < 2$.

(ii) Sketch the curve A given by $y = \ln|x^2 - 4|$.

(iii) Show that the area of the finite region enclosed by the positive x-axis, the y-axis and the curve A is $4\ln(2+\sqrt{3}) - 2\sqrt{3}$.

(iv) The curve B is given by $y = \left|\ln|x^2-4|\right|$. Find the area between the curve B and the x-axis with $|x| < 2$.

[*Note: you may assume that $t \ln t \to 0$ as $t \to 0$.*]

You will need the following standard results (given in your formula booklet) for the main exercise. We'll prove them later. ✐

$$\int \tan x \; dx = \ln|\sec x| + C \qquad \int \sec x \; dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x \; dx = \ln|sin\, x| + C \qquad \int \operatorname{cosec} x \; dx = \ln|\operatorname{cosec} x + \cot x| + C$$

SKILL #7: Using Partial Fractions

We saw earlier that we can split some expressions into partial fractions. This allows us to integrate some expressions with more complicated denominators.

Find $\int \frac{2}{x^2-1}\, dx$

?

Further Examples

Find $\int \frac{x-5}{(x+1)(x-2)}\ dx$

?

Find $\int \frac{8x^2-19x+1}{(2x+1)(x-2)^2}\ dx$

?

Test Your Understanding

Edexcel C4 June 2009 Q3

$$f(x) = \frac{4-2x}{(2x+1)(x+1)(x+3)} = \frac{A}{(2x+1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)}.$$

(*a*) Find the values of the constants A, B and C. **(4)**

(*b*) (i) Hence find $\int f(x)\ dx$. **(3)**

(ii) Find $\int_0^2 f(x)\ dx$ in the form $\ln k$, where k is a constant. **(3)**

(a)

?

(b) (i)

?

(ii)

?

SKILL #8: Integrating top-heavy algebraic fractions

$$\int \frac{x^2}{x+1}\,dx = ?$$

How would we deal with this? (the clue's in the title)

Some manipulation to simplify
?

Now integrate
?

Test Your Understanding

$$\int \frac{x}{x-1}\,dx$$

$=$

$=$

?

$$\int \frac{x^3+2}{x+1}\,dx$$

$=$

?

Contrast this with $\int \frac{x-1}{x}\,dx$ which can be integrated more simply:

$$\int \frac{\boldsymbol{x-1}}{\boldsymbol{x}}\,\boldsymbol{dx} = \int \boldsymbol{1 - \frac{1}{x}}\,\boldsymbol{dx} = \boldsymbol{x - \ln|x| + C}$$

Exercise 11G

Pearson Pure Mathematics Year 2/AS
Page 312-313

Extension:

[MAT 2003 1D]
What is the exact value of the definite integral

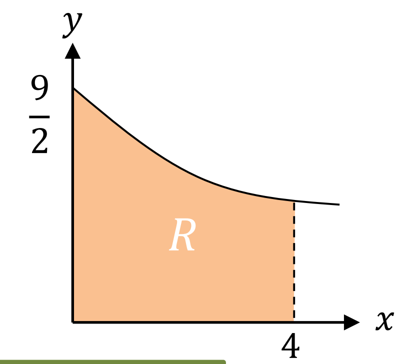
$$\int_1^2 \frac{dx}{x + x^3}$$

Solution: ?

Finding Areas

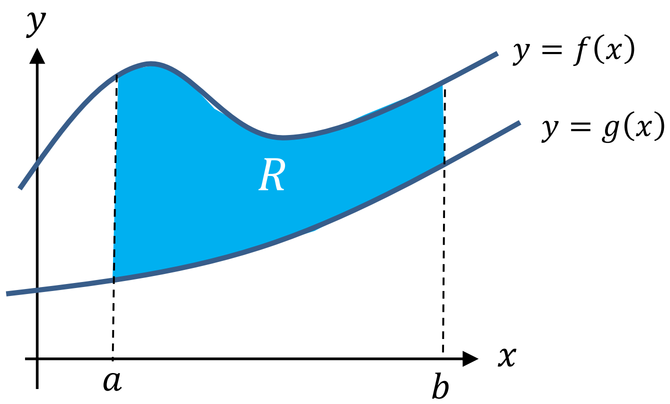
You're already familiar with the idea that definite integration gives you the (signed) area bound between the curve and the x-axis.
Given your expanded integration skills, you can now find the area under a greater variety of curves.

[Textbook] The diagram shows part of the curve $y = \frac{9}{\sqrt{4+3x}}$
The region R is bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$, as shown in the diagram. Use integration to find the area of R.



?

Skill #9: Area between two curves



(This was presented in my Year 1 slides as an 'alternative method')

The areas under the two curves are $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$. It therefore follows the area between them (provided the curves don't overlap) is:

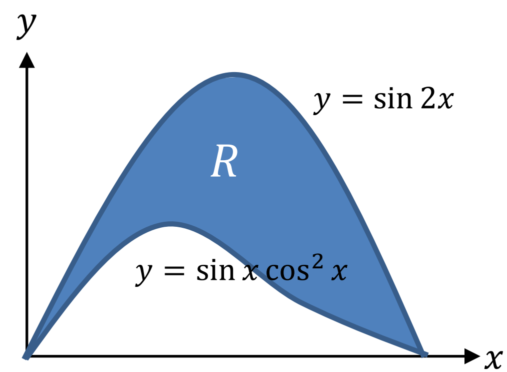
$$R = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b \big(f(x) - g(x)\big)\,dx$$

Fro Tip: Ensure you have top curve minus bottom curve.

[Textbook] The diagram shows part of the curves $y = \sin 2x$ and $y = \sin x \cos^2 x$ where $0 \le x \le \frac{\pi}{2}$. The region R is bounded by the two curves. Use integration to find the area of R.

?



Test Your Understanding

Edexcel C4 Jan 2009 Q2

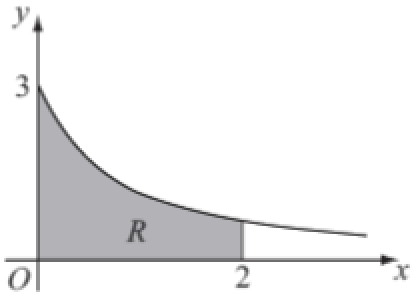


Figure 1

Figure 1 shows part of the curve $y = \dfrac{3}{\sqrt{(1+4x)}}$. The region R is bounded by the curve, the x-axis, and the lines $x = 0$ and $x = 2$, as shown shaded in Figure 1.

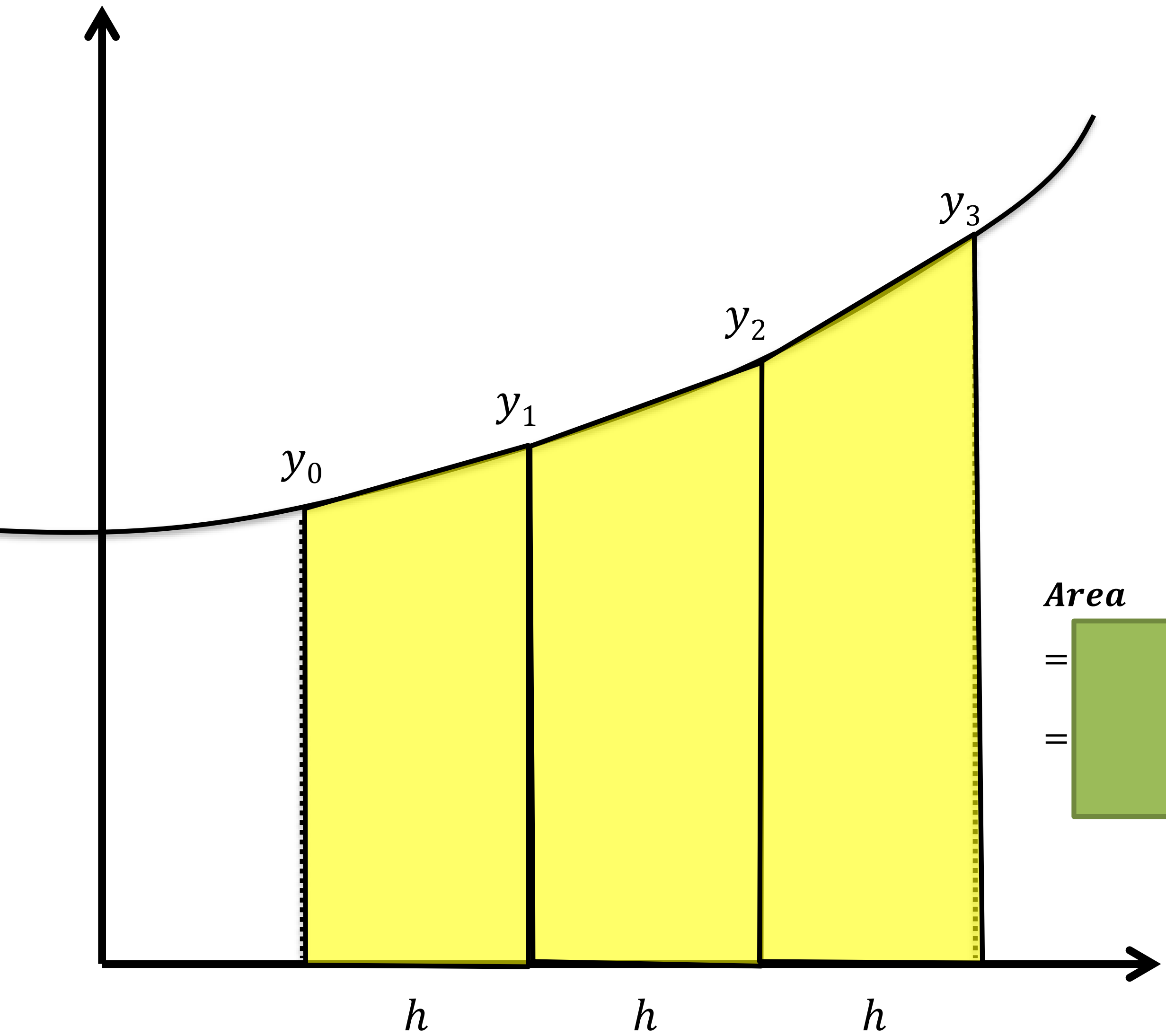
(a) Use integration to find the area of R. **(4)**

?

Exercise 11H

Pearson Pure Mathematics Year 2/AS
Pages 314-317

Skill #10: Trapezium Rule



Sometimes finding the exact area under the graph via integration is difficult. Students who have taken GCSE Maths may be familiar with the idea of **approximating the area under a graph by dividing it into trapeziums of equal width**.

Area

$=$

$=$

?

Trapezium Rule

In general:

width of each trapezium

$$\int_a^b y\,dx \approx \frac{h}{2}\left(y_1 + 2(y_2 + \cdots + y_{n-1}) + y_n\right)$$

Area under curve

is approximately

Example

We're approximating the region bounded between $x = 1$, $x = 3$, the x-axis the curve $y = x^2$, using 4 strips.

x	?
y	?

Dividing a gap of 2 into 4 strips means each strip will be width 0.5

$h =$?

$Area \approx$?

Trapezium Rule

Edexcel C2 May 2013 (R) Q2

$$y = \frac{x}{\sqrt{(1+x)}}$$

(*a*) Complete the table below with the value of y corresponding to $x = 1.3$, giving your answer to 4 decimal places.

(1)

Fro Tip: You can generate table with Casio calcs . $Mode \rightarrow 3$ $(Table)$. Use 'Alpha' button to key in X within the function. Press =

x	1	1.1	1.2	1.3	1.4	1.5
y	0.7071	0.7591	0.8090	?	0.9037	0.9487

(*b*) Use the trapezium rule, with all the values of y in the completed table, to obtain an approximate value for

$$\int_1^{1.5} \frac{x}{\sqrt{(1+x)}}\,dx$$

giving your answer to 3 decimal places.

You must show clearly each stage of your working.

(4)

$Area \approx$?

Further Example

Trapezium Rule: $\int_a^b y\,dx \approx \frac{1}{2}h[y_0 + 2(y_1 + \cdots + y_{n-1}) + y_n]$

Q

Given $I = \int_0^{\frac{\pi}{3}} \sec x \; dx$

a) Find the exact value of I.

b) Use the trapezium rule with two strips to estimate I.

c) Use the trapezium rule with four strips to find a second estimate of I.

d) Find the percentage error in using each estimate.

a ?

b ?

c ?

d ?

Test Your Understanding

Edexcel C4 June 2014(R) Q2

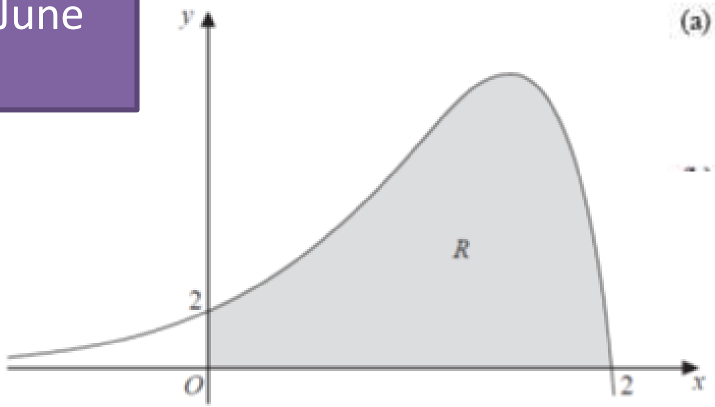


Figure 1

(a)

?

(b)

?

(c)

?

Figure 1 shows a sketch of part of the curve with equation

$$y = (2 - x)e^{2x},$$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis and the y-axis.

The table below shows corresponding values of x and y for $y = (2 - x)e^{2x}$.

x	0	0.5	1	1.5	2
y	2	4.077	7.389	10.043	0

(*a*) Use the trapezium rule with all the values of y in the table, to obtain an approximation for the area of R, giving your answer to 2 decimal places.

(3)

(*b*) Explain how the trapezium rule can be used to give a more accurate approximation for the area of R.

(1)

(*c*) Use calculus, showing each step in your working, to obtain an exact value for the area of R. Give your answer in its simplest form.

(5)

Exercise 11I

Pearson Pure Mathematics Year 2/AS
Pages 319-322

Integration with Parametric Equations

Suppose we have the following parametric equations:

$$x = t^2$$
$$y = t + 1$$

To find the area under the curve, we want to determine to determine $\int y \, dx$.
The problem however is that y is in terms of t, not in terms of x.

Area: $\int y \, dx =$?

Fro Memory Tip: No need to remember the whole new formulae. Just remember that $\frac{dx}{dt} dt = dx$, which follows from the chain rule (and very informally, you can see holds as the dt's cancel)

Determine the area bound between the curve with parametric equations $x = t^2$ and $y = t + 1$, the x-axis, and the lines $x = 0$ and $x = 3$.

$\frac{dx}{dt} =$?

?

?

STEP 1: Find $\frac{dx}{dt}$

STEP 2: Change limits

STEP 3: Integrate

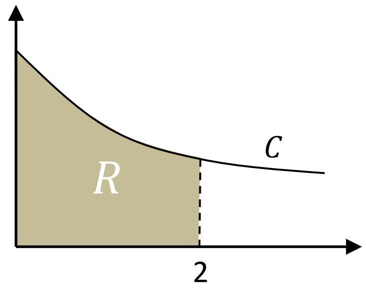
Since we're now integrating in terms of t, we need to change the limits so they're in terms of t.

Further Example

[Textbook] The curve C has parametric equations

$$x = t(1+t), \qquad y = \frac{1}{1+t}, \qquad t \geq 0$$

Find the exact area of the region R, bounded by C, the x-axis and the lines $x = 0$ and $x = 2$.



$\frac{dx}{dt} =$?

When $x = 0$, ?

When $x = 2$, ?

?

Recall that to integrate a top-heavy fraction, use long-division first.

Test Your Understanding

Edexcel C4 Jan 2013 Q5

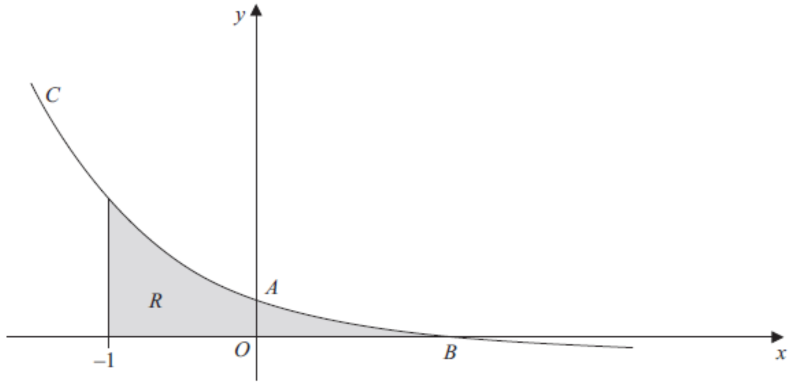


Figure 2 shows a sketch of part of the curve C with parametric equations

$$x = 1 - \frac{1}{2}t, \qquad y = 2^t - 1.$$

The curve crosses the y-axis at the point A and crosses the x-axis at the point B.

(*a*) Show that A has coordinates $(0, 3)$. **(2)**

(*b*) Find the x-coordinate of the point B. **(2)**

The region R, as shown shaded in Figure 2, is bounded by the curve C, the line $x = -1$ and the x-axis.

(*d*) Use integration to find the exact area of R. **(6)**

Helping Hand:

$$\frac{d}{dx}(a^x) = a^x(\ln a)$$

$$\int a^x \; dx = \frac{a^x}{\ln a} + c$$

(d)

Working parametrically:

(a)

?

(b)

?

?

Exercise ?

This exercise is not in the current version of the Pearson textbooks as the content was added later. I have temporarily included the exercise subsequently produced by Pearson.

ANSWERS

1 ?

2 ?

3 ?

4 ?

5 ?

6 ?

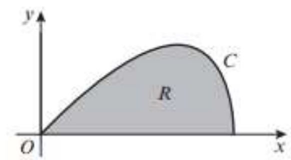
7 ?

(P) **1** The curve C has parametric equations $x = t^3$, $y = t^2$, $t \geqslant 0$. Show that the exact area of the region bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$ is $k\sqrt[3]{2}$, where k is a rational constant to be found.

(E/P) **2** The curve C has parametric equations

$$x = \sin t,\ y = \sin 2t,\ 0 \geqslant t \geqslant \frac{\pi}{2}$$

The finite region R is bounded by the curve and the x-axis. Find the exact area of R. **(6 marks)**



(E/P) **3** This graph shows part of the curve C with parametric equations $x = (t + 1)^2$, $y = \frac{1}{2}t^3 + 3$, $t \geqslant -1$

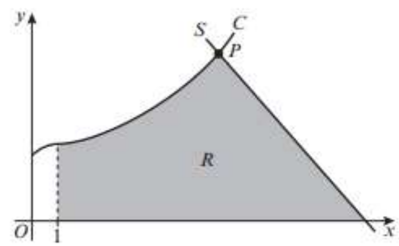
P is the point on the curve where $t = 2$.

The line S is the normal to C at P.

a Find an equation of S. **(5 marks)**

The shaded region R is bounded by C, S, the x-axis and the line with equation $x = 1$.

b Using integration, find the area of R. **(5 marks)**

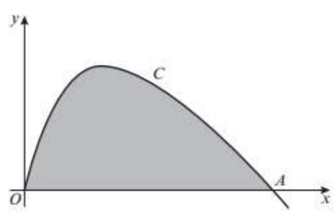


(E/P) **4** The diagram shows the curve C with parametric equations

$$x = 3t^2,\ y = \sin 2t,\ t \geqslant 0.$$

a Write down the value of t at the point A where the curve crosses the x-axis. **(1 mark)**

b Find, in terms of π, the exact area of the shaded region bounded by C and the x-axis. **(6 marks)**



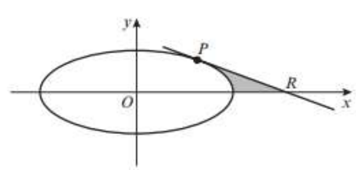
(E/P) **5** The curve shown has parametric equations

$$x = 5\cos\theta,\ y = 4\sin\theta,\ 0 \leqslant \theta < 2\pi$$

a Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$ **(3 marks)**

b Find an equation of the tangent to the curve at the point P. **(3 marks)**

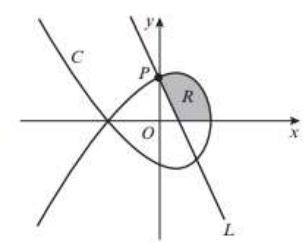
c Find the exact area of the shaded region bounded by the tangent PR, the curve and the x-axis. **(6 marks)**



(E/P) **6** The curve C has parametric equations

$$x = 1 - t^2,\ y = 2t - t^3,\ t \in \mathbb{R}$$

The line L is a normal to the curve at the point P where the curve intersects the positive y-axis. Find the exact area of the region R bounded by the curve C, the line L and the x-axis, as shown on the diagram. **(7 marks)**



(E/P) **7** The curve shown in the diagram has parametric equations

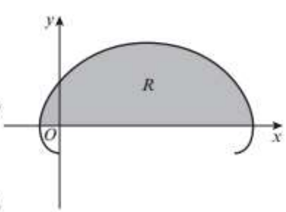
$$x = t - 2\sin t,\ y = 1 - 2\cos t,\ 0 \leqslant t \leqslant 2\pi$$

a Show that the curve crosses the x-axis where $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$ **(3 marks)**

The finite region R is enclosed by the curve and the x-axis, as shown shaded in the diagram.

b Show that the area R is given by $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2\, dt$ **(3 marks)**

c Use this integral to find the exact value of the shaded area. **(4 marks)**



SKILL #11: Differential Equations (We're on the home straight!)

Differential equations are equations involving a mix of variables and derivatives, e.g. y, x and $\frac{dy}{dx}$.

'Solving' these equations means to get y in terms of x (with no $\frac{dy}{dx}$).

Q Find the general solution to $\frac{dy}{dx} = xy + y$

?

?

?

STEP 1: Get y to the side of $\frac{dy}{dx}$ by dividing and x to the other side.
(you may need to factorise to separate out y first)

STEP 2: Integrate both sides with respect to x. $\frac{dy}{dx}\,dx$ simplifies to dy (recall that (implicitly) differentiating an expression in terms of y with respect to x introduces a $\frac{dy}{dx}$, so integrating similarly would get rid of it)

STEP 3: Make y the subject, if the question asks.

Fro Tip: When you have $e^{f(x)+c} = e^{f(x)}e^c$, it's typical to write $Ae^{f(x)}$ where $A = e^c$ (noting that A is any positive constant since $e^c > 0$)

Another Example

Q Find the general solution to $(1+x^2)\frac{dy}{dx} = x\tan y$

?

STEP 1: Get y to the side of $\frac{dy}{dx}$ by dividing and x to the other side.
(you may need to factorise to separate out y first)

?

STEP 2: Integrate both sides with respect to x.

?

STEP 2b: If possible, try to combine your constant of integration with other terms
(e.g. by letting $C = \ln k$ where k is another constant)

?

STEP 3: Make y the subject, if the question asks.

Differential Equations with Boundary Conditions

Q [Textbook] Find the general solution to $\frac{dy}{dx} = -\frac{3(y-2)}{(2x+1)(x+2)}$

Given that $x = 1$ when $y = 4$. Leave your answer in the form $y = f(x)$

?

Test Your Understanding

Edexcel C4 Jan 2012 Q4

Given that $y = 2$ at $x = \frac{\pi}{4}$, solve the differential equation

$$\frac{dy}{dx} = \frac{3}{y\cos^2 x}. \qquad \textbf{(5)}$$

?

Key Points on Differential Equations

- Get y on to LHS by dividing (possibly factorising first).
- If after integrating you have ln on the RHS, make your constant of integration $\ln k$.
- Be sure to combine all your ln's together just as you did in C2. E.g.:

$$2\ln|x+1| - \ln|x| \quad \rightarrow \quad ?$$

- Sub in boundary conditions to work out your constant – better to do sooner rather than later.
- Exam questions ♥ partial fractions combined with differential equations.

Exercise 11J

Pearson Pure Mathematics Year 2/AS
Pages 324-326

Extension:

(STEP I 2013 Q7)

(i) Use the substitution $y = ux$, where u is a function of x, to show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$ is

$$y = x\sqrt{4 + 2\ln x} \qquad (x > \mathrm{e}^{-2}).$$

(ii) Use a substitution to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

(iii) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

Solutions: ?

Forming differential equations

Differential equations are useful because regularly in real-life, the rate of change of a variable is based on its current value. For example in Year 1, we saw a property of exponential growth is that the **rate of change is proportional to the current value**:

Q The rate of increase of a rabbit population (with population P, where time is t) is **proportional to** the current population.
Form a differential equation, and find its general solution.

? Form equation

? Solve it

(Notice by the way that $e^{kt} = \left(e^k\right)^t$, and since e^k is a constant, we could always write $y = A \cdot B^x$. i.e. The general solution is 'any generic exponential function', not just restricted to those with e as the base. However it is customary to write Ae^{kt})

Further Example

[Textbook] Water in a manufacturing plant is held in a large cylindrical tank of diameter 20m. Water flows out of the bottom of the tank through a tap at a rate proportional to the cube root of the volume.

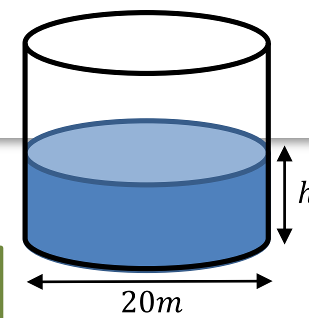
(a) Show that t minutes after the tap is opened, $\frac{dh}{dt} = -k\sqrt[3]{h}$ for some constant k.

(b) Show that the general solution of this differential equation may be written $h = (P - Qt)^{\frac{3}{2}}$, where P and Q are constants.

Initially the height of the water is 27m. 10 minutes later, the height is 8m.

(c) Find the values of the constants P and Q.

(d) Find the time in minutes when the water is at a depth of 1m.



a ?

b ?

c ?

d ?

Test Your Understanding

Edexcel C4 June 2005 Q8

Liquid is pouring into a container at a constant rate of 20 $cm^3\ s^{-1}$ and is leaking out at a rate proportional to the volume of the liquid already in the container.

(*a*) Explain why, at time t seconds, the volume, V cm^3, of liquid in the container satisfies the differential equation

$$\frac{dV}{dt} = 20 - kV,$$

where k is a positive constant.

The container is initially empty.

(*b*) By solving the differential equation, show that

$$V = A + Be^{-kt},$$

giving the values of A and B in terms of k. **(6)**

Given also that $\frac{dV}{dt} = 10$ when $t = 5$,

(*c*) find the volume of liquid in the container at 10 s after the start. **(5)**

(a)

?

Teachers/Students: I recommend also looking at Edexcel Jan 2008 Q8 which has a part (a) similar to the previous example.

(b)

?

(6)

:)

?

Exercise 11K

Pearson Pure Mathematics Year 2/AS
Pages 328-329

Extension:
[STEP 2011 Q7]

In this question, you may assume that $\ln(1+x) \approx x - \frac{1}{2}x^2$ when $|x|$ is small.

The height of the water in a tank at time t is h. The initial height of the water is H and water flows into the tank at a constant rate. The cross-sectional area of the tank is constant.

(i) Suppose that water leaks out at a rate proportional to the height of the water in the tank, and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(\alpha^2 H - h)\,,$$

for some positive constant k. Deduce that the time T taken for the water to reach height αH is given by

$$kT = \ln\left(1 + \frac{1}{\alpha}\right),$$

and that $kT \approx \alpha^{-1}$ for large values of α.

(ii) Suppose that the rate at which water leaks out of the tank is proportional to $\sqrt{h}$ (instead of h), and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that the time T' taken for the water to reach height αH is given by

$$cT' = 2\sqrt{H}\left(1 - \sqrt{\alpha} + \alpha \ln\left(1 + \frac{1}{\sqrt{\alpha}}\right)\right)$$

for some positive constant c, and that $cT' \approx \sqrt{H}$ for large values of α.

Summary of Functions

$f(x)$	How to deal with it	$\int f(x)dx$ (+constant)	Formula booklet?
$\sin x$	?		No
$\cos x$	?		No
$\tan x$	?		Yes
$\sin^2 x$	?		No
$\cos^2 x$	?		No
$\tan^2 x$	?		No
$cosec\ x$	?		Yes
$\sec x$	?		Yes
$\cot x$	?		Yes

Summary of Functions

$f(x)$	How to deal with it	$\int f(x)dx$ (+constant)	Formula booklet?
$cosec^2 x$	?		No!
$sec^2 x$	?		Yes (but memorise)
$cot^2 x$	?		No
$\sin 2x \cos 2x$	?		No
$\frac{1}{x}$	?		No
$\ln x$	?		No
$\frac{x}{x+1}$	?		
$\frac{1}{x(x+1)}$	?		

Summary of Functions

$f(x)$	How to deal with it	$\int f(x) dx$ (+constant)
$\dfrac{4x}{x^2+1}$	?	
$\dfrac{x}{(x^2+1)^2}$	?	
e^{2x+1} $\dfrac{1}{1-3x}$	?	
$x\sqrt{2x+1}$	?	
$\sin^5 x \; \cos x$	?	