P1 Chapter 7 :: Algebraic Methods

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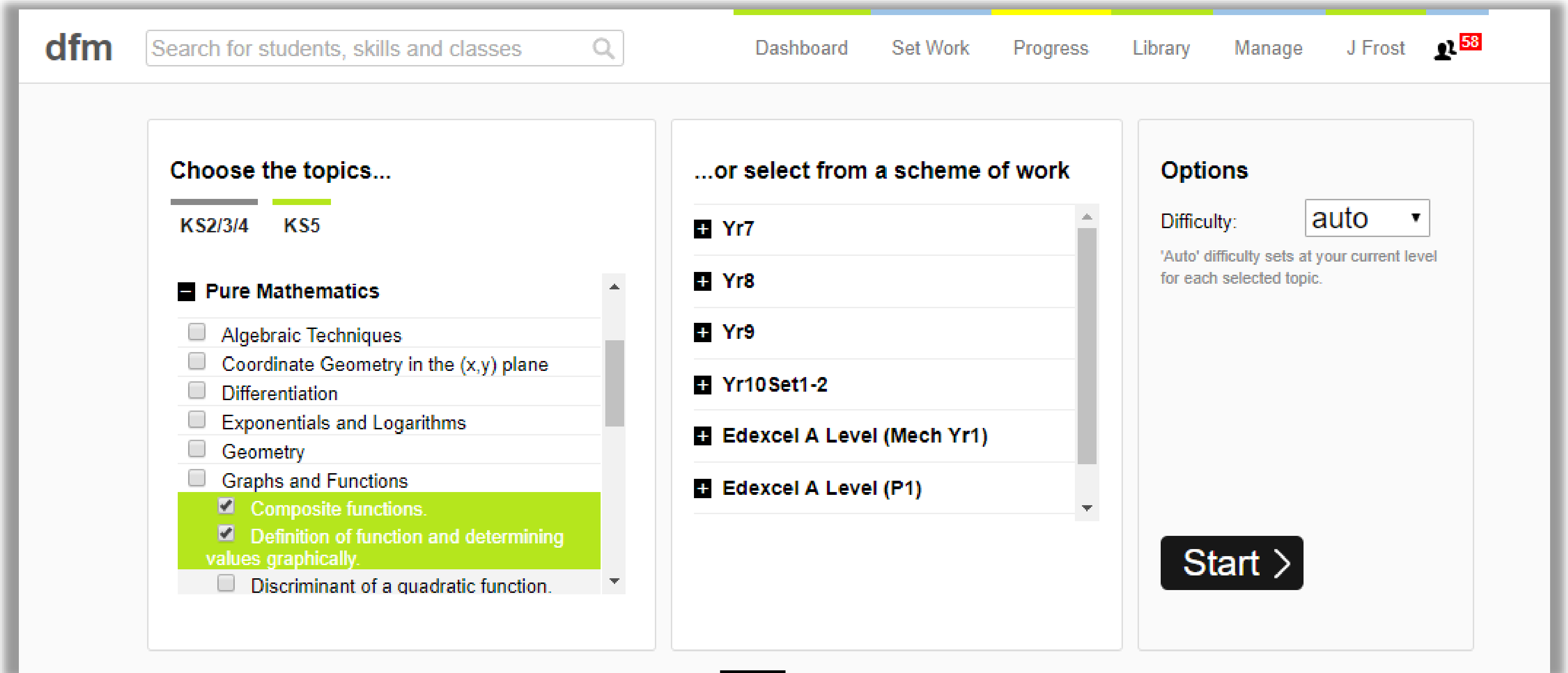

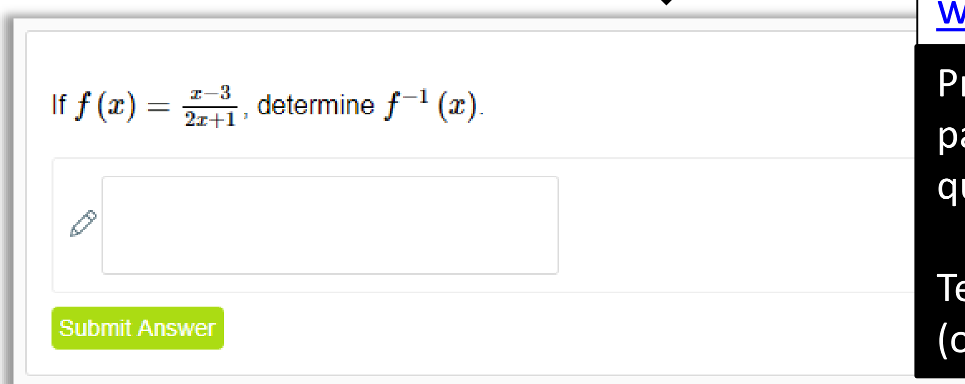

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Chapter Overview

This chapter covers a number of algebraic techniques, but also algebraic 'proofs'.

1:: Algebraic Fractions

Simplify $\frac{2x^2-5x-3}{2x^2-9x+9}$

2:: Dividing Polynomials

Divide $x^3 + 2x^2 - 17x + 6$

3:: The Factor Theorem

Given that $(x-1)$ is a factor of $5x^3 - 9x^2 + 2x + a$, find the value of a.

A Level 2017 specification note: The 'Remainder Theorem' has been removed (it was included in C2). Sad times…

4:: Proof

Prove that all square numbers are either a multiple of 4 or 1 more than a multiple of 4.

1 :: Simplifying Algebraic Fractions

Recall that you can simplify fractions by **dividing** the numerator and denominator by a **common factor**.

Hint: To identify common factors we need to factorise first.

$$\frac{x^2-1}{x^2+x} = ?$$

Note: Do not leave 1 in the denominator!

$$\frac{x^2+3x+2}{x+1} = ?$$

Factorise the easier one first because it provides clues to the factorisation of the other.

$$\frac{2x^2+11x+12}{x^2+9x+20} = ?$$

$$\frac{4-x^2}{x^2+2x-8} = ?$$

Tip: $\frac{a-b}{b-a} = -1$

Exercise 7A

Pearson Pure Mathematics Year 1/AS
Pages 138-139

Long Division: Terminology

$$11 \div 4 = 2\ rem\ 3$$

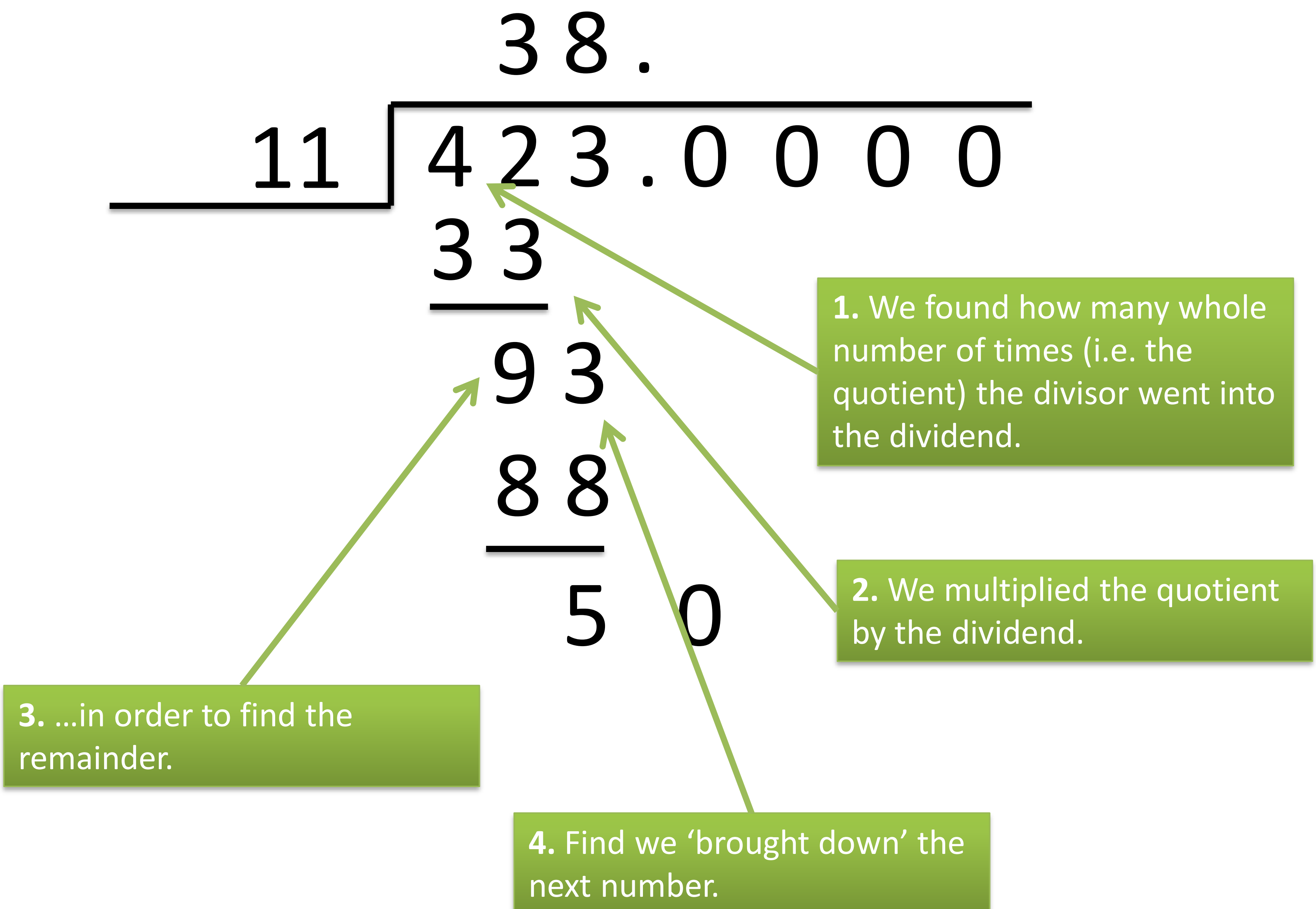
?

?

?

?

Normal Long Division



How many times does x go into $6x^3$?

Now repeat! How many times does x go into $-2x^2$?

$$
\begin{array}{r|l}
 & 6x^2 - 2x + 3 \\
\hline
x+5 & 6x^3 + 28x^2 - 7x + 15 \\
 & 6x^3 + 30x^2 \\
\hline
 & \quad -2x^2 - 7x \\
 & \quad -2x^2 - 10x \\
\hline
 & \qquad\qquad 3x + 15 \\
 & \qquad\qquad 3x + 15 \\
\hline
 & \qquad\qquad\qquad 0
\end{array}
$$

Multiply $6x^2$ by $(x+5)$. The first term should match with above.

Subtract and carry down next term.

Tip:
You can check your solution by expanding:
$$(x+5)(6x^2-2x+3)$$
But if you know you should get no remainder, ending with 0 at the bottom is a good sign!

Tip:
Be <u>**very**</u> careful subtracting negatives. $-7x-(-10x)=3x$

This is the remainder.

Further Example

Find the remainder when $3x^3 - 2x + 4$ is divided by $x - 1$.

$$
\begin{array}{r|l}
 & 3x^2 + 3x + 1 \\
\hline
x - 1 & 3x^3 + 0x^2 - 2x + 4 \\
 & \underline{3x^3 - 3x^2} \\
 & 3x^2 - 2x \\
 & \underline{3x^2 - 3x} \\
 & x + 4 \\
 & \underline{x - 1} \\
 & 5
\end{array}
$$

Note: Ensure that any missing terms in the polynomial are filled in, so that the powers decrease by 1 each time.

The remainder is 5.

Test Your Understanding

Find the remainder when $2x^3 - 5x^2 - 16x + 10$ is divided by $x - 4$.

?

Further Test Your Understanding

Divide $8x^3 - 1$ by $2x - 1$.

?

Exercise 7B

Pearson Pure Mathematics Year 1/AS
Pages 141-142

3 :: The Factor Theorem

$$x^3 + x^2 - 4x - 4 = (x - 2)(x^2 + 3x + 2)$$

We can see that $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$.
What would happen if x is 2?

?

The Factor Theorem states that if $f(x)$ is a polynomial then:

- If $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$.
- Conversely, if $(x - p)$ is a factor of $f(x)$, then $f(p) = 0$.

Examples (in textbook)

Show that $(x-2)$ is a factor of $x^3 + x^2 - 4x - 4$.

?

Fully factorise $2x^3 + x^2 - 18x - 9$.

?

Using Factor Theorem to find unknown coefficients

Given that $2x + 1$ is a factor of $6x^3 + ax^2 + 1$, determine the value of a.

?

Test Your Understanding

Edexcel C2 May 2016 Q2

$$f(x) = 6x^3 + 13x^2 - 4$$

No long in spec.

~~(a) Use the remainder theorem to find the remainder when f(x) is divided by (2x + 3). (2)~~

(b) Use the factor theorem to show that $(x + 2)$ is a factor of $f(x)$. **(2)**

(c) Factorise $f(x)$ completely. **(4)**

?

Given that $3x - 1$ is a factor of $3x^3 + 11x^2 + ax + 1$, determine the value of a.

?

Exercise 7C

Pearson Pure Mathematics Year 1/AS
Pages 145-146

Extension

1 *[MAT 2006 1E]* The cubic
$$x^3 + ax + b$$
Has both $(x-1)$ and $(x-2)$ as factors.
Determine the values of a and b.

?

2 [MAT 2009 1I] The polynomial
$$n^2x^{2n+3} - 25nx^{n+1} + 150x^7$$
Has $x^2 - 1$ as a factor

A) for no values of n;
B) for $n = 10$ only;
C) for $n = 15$ only;
D) for $n = 10$ and $n = 15$ only.

?

The **remainder theorem** states that if $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$. This similarly works whenever a makes the divisor 0.

3 [MAT 2013 1G] Let $n \geq 2$ be an integer and $p_n(x)$ be the polynomial
$$p_n(x) = (x-1) + (x-2) + \cdots + (x-n)$$
What is the remainder, in terms of n, when $p_n(x)$ is divided by $p_{n-1}(x)$?

?

4 :: Proof

Terminology

A **conjecture** is a mathematical statement that has yet to be proven.

One famous conjecture is **Goldbach's Conjecture**.
It states "*Every even integer greater than 2 can be expressed as the sum of two primes*."
It has been verified up to 4×10^{18} (that's big!); this provides evidence that it is true, but does not prove it is true!

A **theorem** is a mathematical statement that has been proven.

One famous misnomer was **Fermat's Last Theorem**, which states "*If* n is an integer where $n > 2$, *then* $a^n + b^n = c^n$ *has no non-zero integer solutions for* a, b, c." It was 300 years until this was proven in 1995. Only then was the 'Theorem' in the name then correct!

Types of Proof

A proof must show all **assumptions** you are using, have a clear **sequential list of steps** that logically follow, and must cover **all possible cases**.
You should usually make a **concluding statement**, e.g. restating the original conjecture that you have proven.

a. Proof by Deduction

This is the simplest type, where you start from known facts and reach the desired conclusion via deductive steps.

An **identity** is an equation that is true for **all values** of the variables. e.g. $x^2 = 4$ is true only for $x = \pm 2$, but $x(x-1) \equiv x^2 - x$ is true for all x.

"Prove that the product of two odd numbers is odd."

?

"Prove that $(3x+2)(x-5)(x+7) \equiv 3x^3 + 8x^2 - 101x - 70$"

?

Be Warned…

Proof by Deduction requires you to **start from known facts** and end up at the conclusion. It is **not** acceptable to start with to the conclusion, and verify it works, **because you are assuming the thing you are trying to prove**.

Example: Prove that if three consecutive integers are the sides of a right-angled triangle, they must be 3, 4 and 5.

Incorrect Proof:

?

We are assuming the thing we are trying to prove!

Correct Proof:

?

We are only assuming things in the 'if' bit. This is fine!

The underlying problem is that this technique doesn't prove there can't be **other** consecutive integers that work – we have only verified 3,4,5 is one such solution.

Types of Proof

a. Proof by Deduction

Exam Tip: This is quite a common last parter.

Prove that $x^2 + 4x + 5$ is positive for all values of x.

?

Anything squared is at least 0. This is formally known as the '*trivial inequality*'.

Test Your Understanding

Prove that the sum of the squares of two consecutive odd numbers is 2 more than a multiple of 8.

?

Exercise 7D

Pearson Pure Mathematics Year 1/AS
Pages 149-150

Extension

[STEP I 2005 Q1] 47231 is a five-digit number whose digits sum to $4 + 7 + 2 + 3 + 1 = 17$.

(i) Prove that there are 15 five-digit numbers whose digits sum to 43. You should explain your reasoning clearly.

(ii) How many five-digit numbers are there whose digits sum to 39?

? i

? ii

Other Types of Proof

b. Proof by Exhaustion

This means breaking down the statement into **all possible smaller cases**, where we prove each individual case.
(This technique is sometimes known as 'case analysis')

n is either even or odd.

?

c. Disproof by Counter-Example

While to prove a statement is true, we need to prove every possible case (potentially infinitely many!), **we only need one example to disprove** a statement.
This is known as a **counterexample**.

Disprove the statement:
"$n^2 - n + 41$ is prime for all integers n."

?

Another Type in the A Level Specification (Year 2)

c. Proof by Contradiction

We assume the original statement is false, and show this results in a contradiction.

Prove that $\sqrt{2}$ is irrational.

?

Exercise 7E

Pearson Pure Mathematics Year 1/AS
Pages 152-154

Extension

[STEP I 2008 Q1] What does it mean to say that a number x is irrational?
Prove by contradiction statements A and B below, where p and q are real numbers.

A: If pq is irrational, then at least one of p and q is irrational.
B: If $p + q$ is irrational, then at least one of p and q is irrational.

Disprove by means of a counterexample statement C below, where p and q are real numbers.

C: If p and q are irrational, then $p + q$ is irrational.

Note: e is Euler's Number and you will encounter it later at A Level. But for the moment, you only need to know it is an irrational number, like π.

If the numbers e, π, π^2, e^2 and $e\pi$ are irrational, prove that at most one of the numbers $\pi + e, \pi - e, \pi^2 - e^2, \pi^2 + e^2$ is rational.

<u>**Solutions on next slide.**</u>

Solution to Extension Problem

What does it mean to say that a number x is irrational?

?

Prove by contradiction statements A and B below, where p and q are real numbers.

A: *If pq is irrational, then at least one of p and q is irrational.*

B: *If $p + q$ is irrational, then at least one of p and q is irrational.*

?

Disprove by means of a counterexample statement C below, where p and q are real numbers.

C: *If p and q are irrational, then $p + q$ is irrational.*

?

Solution to Extension Problem

If the numbers e, π, π^2, e^2 *and* $e\pi$ *are irrational, prove that at most one of the numbers* $\pi + e$, $\pi - e$, $\pi^2 - e^2$, $\pi^2 + e^2$ *is rational.*

?

Fro Fun Fact: No mathematician has yet managed to prove that $\pi + e$ or $\pi - e$ is irrational.