P2 Chapter 7 :: Trigonometry And Modelling

Chapter Overview

This chapter is very similar to the trigonometry chapters in Year 1. The only difference is that new trig functions: sec, $cosec$ and cot, are introduced.

1a:: Addition Formulae

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

1b:: Double Angle Formulae

"Solve, for $0 \leq x < 2\pi$, the equation

$$2\tan 2y \tan y = 3$$

giving your solutions to 3sf."

3:: Simplifying $a\cos x \pm b\sin x$

"Find the maximum value of $2\sin x + \cos x$ and the value of x for which this maximum occurs."

4:: Modelling

"The sea depth of the tide at a beach can be modelled by $x = Rsin\left(\frac{2\pi t}{5} + \alpha\right)$, where t is the hours after midnight..."

Topics	What students need to learn:		
	Content		Guidance
5 **Trigonometry** *continued*	5.5	**Understand and use** $\tan\theta = \frac{\sin\theta}{\cos\theta}$ **Understand and use** $\sin^2\theta + \cos^2\theta = 1$ $\sec^2\theta = 1 + \tan^2\theta$ and $\text{cosec}^2\theta = 1 + \cot^2\theta$	**These identities may be used to solve trigonometric equations** and angles may be in degrees or radians. **They may also be used to prove further identities.**
	5.6	Understand and use double angle formulae; use of formulae for $\sin(A \pm B)$, $\cos(A \pm B)$, and $\tan(A \pm B)$, understand geometrical proofs of these formulae. Understand and use expressions for $a\cos\theta + b\sin\theta$ in the equivalent forms of $r\cos(\theta \pm \alpha)$ or $r\sin(\theta \pm \alpha)$	To include application to half angles. Knowledge of the $\tan(\frac{1}{2}\theta)$ formulae will *not* be required. Students should be able to solve equations such as $a\cos\theta + b\sin\theta = c$ in a given interval.
	5.7	**Solve simple trigonometric equations in a given interval, including quadratic equations in sin, cos and tan and equations involving multiples of the unknown angle.**	**Students should be able to solve equations such as** $\sin(x + 70°) = 0.5$ **for** $0 < x < 360°$, $3 + 5\cos 2x = 1$ **for** $-180° < x < 180°$ $6\cos^2 x + \sin x - 5 = 0, 0 \leqslant x < 360°$ These may be in degrees or radians and this will be specified in the question.
	5.8	Construct proofs involving trigonometric functions and identities.	Students need to prove identities such as $\cos x \cos 2x + \sin x \sin 2x \equiv \cos x$.
	5.9	Use trigonometric functions to solve problems in context, including problems involving vectors, kinematics and forces.	Problems could involve (for example) wave motion, the height of a point on a vertical circular wheel, or the hours of sunlight throughout the year. Angles may be measured in degrees or in radians.

Addition Formulae

Addition Formulae allow us to deal with a **sum or difference of angles**.

How to memorise:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

First notice that for all of these the first thing on the RHS is the same as the first thing on the LHS!

- For sin, the operator in the middle is the same as on the LHS.
- For cos, it's the opposite.
- For tan, it's the same in the numerator, opposite in the denominator.

- For sin, we mix sin and cos.
- For cos, we keep the cos's and sin's together.

Do I need to memorise these? They're all technically in the formula booklet, but you REALLY want to eventually memorise these (particularly the sin and cos ones).

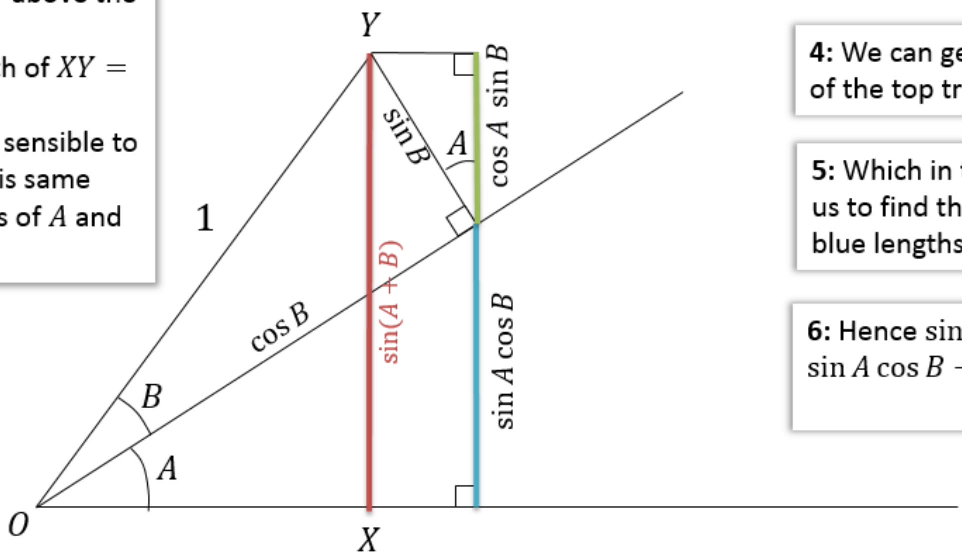
Proof of $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$

(Not needed for exam)

1: Suppose we had a line of length 1 projected an angle of $A + B$ above the horizontal.
Then the length of $XY = \sin(A + B)$
It would seem sensible to try and find this same length in terms of A and B individually.

2: We can achieve this by forming two right-angled triangles.

3: Then we're looking for the combined length of these two lines.



4: We can get the lengths of the top triangle...

5: Which in turn allows us to find the green and blue lengths.

6: Hence $\sin(A + B) = \sin A \cos B + \cos A \sin B$ □

Proof of other identities

Edexcel C3 Jan 2012 Q8

(a) Starting from the formulae for $\sin(A+B)$ and $\cos(A+B)$, prove that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(4)

Examples

Q Given that $2\sin(x + y) = 3\cos(x - y)$ express $\tan x$ in terms of $\tan y$.

Textbook Exercise 7A Page 144

Uses of Addition Formulae

Q [Textbook] Using a suitable angle formulae, show that $\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4}$.

Q [Textbook] Given that $\sin A = -\frac{3}{5}$ and $180° < A < 270°$, and that $\cos B = -\frac{12}{13}$, and B is obtuse, find the value of: (a) $\cos(A - B)$ (b) $\tan(A + B)$

Tip: You can get cos in terms of sin and vice versa by using a rearrangement of $\sin^2 x + \cos^2 x \equiv 1$.
So $\cos A = \sqrt{1 - \sin^2 A}$

Q

Given that $\sin A = -\frac{3}{5}$ and $180° < A < 270°$, and that $\cos B = -\frac{12}{13}$, find the value of: (b) $\tan(A+B)$

Test Your Understanding

Without using a calculator, determine the exact value of:

a) $\cos(75°)$

b) $\tan(75°)$

Challenging question

Edexcel June 2013 Q3

Given that

$$2\cos(x+50)^\circ = \sin(x+40)^\circ.$$

(*a*) Show, without using a calculator, that

$$\tan x^\circ = \frac{1}{3}\tan 40^\circ.$$

Textbook Exercise 7B Page 172

Double Angle Formulae

Double-angle formula allow you to halve the angle within a trig function.

$$\sin(2A) \equiv 2\sin A\cos A$$
$$\cos(2A) \equiv \cos^2 A - \sin^2 A$$
$$\equiv 2\cos^2 A - 1$$
$$\equiv 1 - 2\sin^2 A$$

This first form is relatively rare.

Tip: The way I remember what way round these go is that the cos on the RHS is 'attracted' to the cos on the LHS, whereas the sin is pushed away.

$$\tan(2A) = \frac{2\tan A}{1-\tan^2 A}$$

These are all easily derivable by just setting $A = B$ in the compound angle formulae. e.g.

$$\sin(2A) = \sin(A + A)$$
$$= \sin A\cos A + \cos A\sin A$$
$$= 2\sin A\cos A$$

Examples

[Textbook] Use the double-angle formulae to write each of the following as a single trigonometric ratio.

a) $\cos^2 50° - \sin^2 50°$

b) $\frac{2\tan\left(\frac{\pi}{6}\right)}{1-\tan^2\left(\frac{\pi}{6}\right)}$

c) $\frac{4\sin 70°}{\sec 70°}$

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

Examples

[Textbook] Given that $x = 3\sin\theta$ and $y = 3 - 4cos2\theta$, eliminate θ and express y in terms of x.

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Note: This question is an example of turning a set of **parametric equations** into a single **Cartesian** one. You will cover this in the next chapter.

Given that $\cos x = \frac{3}{4}$ and x is acute, find the exact value of
(a) $\sin 2x$ (b) $\tan 2x$

Textbook Exercise 7C Page 175

Solving Trigonometric Equations

This is effectively the same type of question you encountered in Chapter 6 and in Year 1, except you may need to use either the **addition formulae or double angle formulae.**

[Textbook] Solve $3\cos 2x - \cos x + 2 = 0$ for $0 \leq x \leq 360°$.

Further Examples

[Textbook] By noting that $3A = 2A + A$, :

a) Show that $\sin(3A) = 3\sin A - 4\sin^3 A$.

b) Hence or otherwise, solve, for $0 < \theta < 2\pi$, the equation $16\sin^3\theta - 12\sin\theta - 2\sqrt{3} = 0$

Fro Exam Note: A question pretty much just like this came up in an exam once.

[Textbook] Solve $4\cos(\theta - 30°) = 8\sqrt{2}\sin\theta$ in the range $0 \leq \theta < 360°$.

Test Your Understanding

Edexcel C3 Jan 2013 Q6

6. (i) Without using a calculator, find the exact value of

$$(\sin 22.5° + \cos 22.5°)^2.$$

You must show each stage of your working.

(5)

(ii) (*a*) Show that $\cos 2\theta + \sin \theta = 1$ may be written in the form

$$k \sin^2 \theta - \sin \theta = 0, \text{ stating the value of } k.$$

(2)

(*b*) Hence solve, for $0 \leq \theta < 360°$, the equation

$$\cos 2\theta + \sin \theta = 1.$$

(4)

[Textbook] Solve $2\tan 2y \tan y = 3$ for $0 \leq y < 2\pi$, giving your answer to 2dp.

Textbook Exercise 7D Page 180

$a \sin\theta + b \cos\theta$

Q Put $3\sin x + 4\cos x$ in the form $R\sin(x+\alpha)$ giving α in degrees to 1dp.

STEP 1: Expanding:

STEP 2: Comparing coefficients:

STEP 3: Using the fact that $R^2 \sin^2\alpha + R^2\cos^2\alpha = R^2$:

STEP 4: Using the fact that $\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha$:

STEP 5: Put values back into original expression.

Test Your Understanding

Q Put $\sin x + \cos x$ in the form $R\sin(x+\alpha)$ giving α in terms of π.

Q Put $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x-\alpha)$ giving α in terms of π.

Q Put $2\cos\theta + 5\sin\theta$ in the form $R\cos(\theta - \alpha)$ where $0 < \alpha < 90°$
Hence solve, for $0 < \theta < 360$, the equation $2\cos\theta + 5\sin\theta = 3$

Tip: This is an exam favourite!

Q (Without using calculus), find the maximum value of $12\cos\theta + 5\sin\theta$, and give the smallest positive value of θ at which it arises.

Expression	Maximum	(Smallest) θ at max
$20\sin\theta$		
$5-10\sin\theta$		
$3\cos(\theta+20°)$		
$\dfrac{2}{10+3\sin(\theta-30)}$		

Further Test Your Understanding

Edexcel C3 Jan 2013 Q4

4. (*a*) Express $6\cos\theta + 8\sin\theta$ in the form $R\cos(\theta-\alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$.

Give the value of α to 3 decimal places.

(4)

(*b*) $$\mathrm{p}(\theta) = \frac{4}{12+6\cos\theta+8\sin\theta}, \quad 0 \le \theta \le 2\pi.$$

Calculate

(i) the maximum value of $\mathrm{p}(\theta)$,

(ii) the value of θ at which the maximum occurs.

(4)

Textbook Exercise 7E Page 184

Proving Trigonometric Identities

Just like Chapter 6 had 'provey' and 'solvey' questions, we also get the 'provey' questions in Chapter 7. Just use the appropriate double angle or addition formula.

Prove that $\tan 2\theta \equiv \frac{2}{\cot\theta - \tan\theta}$

Prove that $\frac{1-\cos 2\theta}{\sin 2\theta} \equiv \tan\theta$

Test Your Understanding

[OCR] Prove that $\cot 2x + cosec\ 2x \equiv \cot x$

[OCR] By writing $\cos x = \cos\left(2 \times \frac{x}{2}\right)$ or otherwise, prove the identity $\frac{1-\cos x}{1+\cos x} \equiv \tan^2\left(\frac{x}{2}\right)$

Textbook Exercise 7F Page 187

Very Challenging Exam Example

Edexcel C3 June 2015 Q8

(a) Prove that

$$\sec 2A + \tan 2A \equiv \frac{\cos A + \sin A}{\cos A - \sin A}, \qquad A \neq \frac{(2n+1)\pi}{4}, \ n \in \mathbb{Z}$$

(5)

(b) Hence solve, for $0 \leqslant \theta < 2\pi$,

$$\sec 2\theta + \tan 2\theta = \frac{1}{2}$$

Give your answers to 3 decimal places.

(4)

Modelling

[June 2013 (Withdrawn) Q8]

(a) Express $9\cos\theta - 2\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the exact value of R and give the value of α to 4 decimal places. **(3)**

(b) (i) State the maximum value of $9\cos\theta - 2\sin\theta$

(ii) Find the value of θ, for $0 < \theta < 2\pi$, at which this maximum occurs. **(3)**

Ruth models the height H above the ground of a passenger on a Ferris wheel by the equation

$$H = 10 - 9\cos\left(\frac{\pi t}{5}\right) + 2\sin\left(\frac{\pi t}{5}\right)$$

where H is measured in metres and t is the time in minutes after the wheel starts turning.

(c) Calculate the maximum value of H predicted by this model, and the value of t, when this maximum first occurs. Give your answers to 2 decimal places. **(4)**

(d) Determine the time for the Ferris wheel to complete two revolutions. **(2)**

When trigonometric equations are in the form $Rsin(ax \pm b)$ or $Rcos(ax \pm b)$, they can be used to model various things which have an oscillating behaviour, e.g. tides, the swing of a pendulum and sound waves.

Test Your Understanding

[June 2010 Q7] 2. (*a*) Express $2\sin\theta - 1.5\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the value of α to 4 decimal places. **(3)**

(*b*) (i) Find the maximum value of $2\sin\theta - 1.5\cos\theta$.

(ii) Find the value of θ, for $0 \le \theta < \pi$, at which this maximum occurs. **(3)**

Tom models the height of sea water, H metres, on a particular day by the equation

$$H = 6 + 2\sin\left(\frac{4\pi t}{25}\right) - 1.5\cos\left(\frac{4\pi t}{25}\right), \quad 0 \le t < 12,$$

where t hours is the number of hours after midday.

(*c*) Calculate the maximum value of H predicted by this model and the value of t, to 2 decimal places, when this maximum occurs. **(3)**

(*d*) Calculate, to the nearest minute, the times when the height of sea water is predicted, by this model, to be 7 metres. **(6)**

Tip: Reflect carefully on the substitution you use to allow (bii) to match your identity in (a). θ =?

Textbook Exercise 7G Page 190