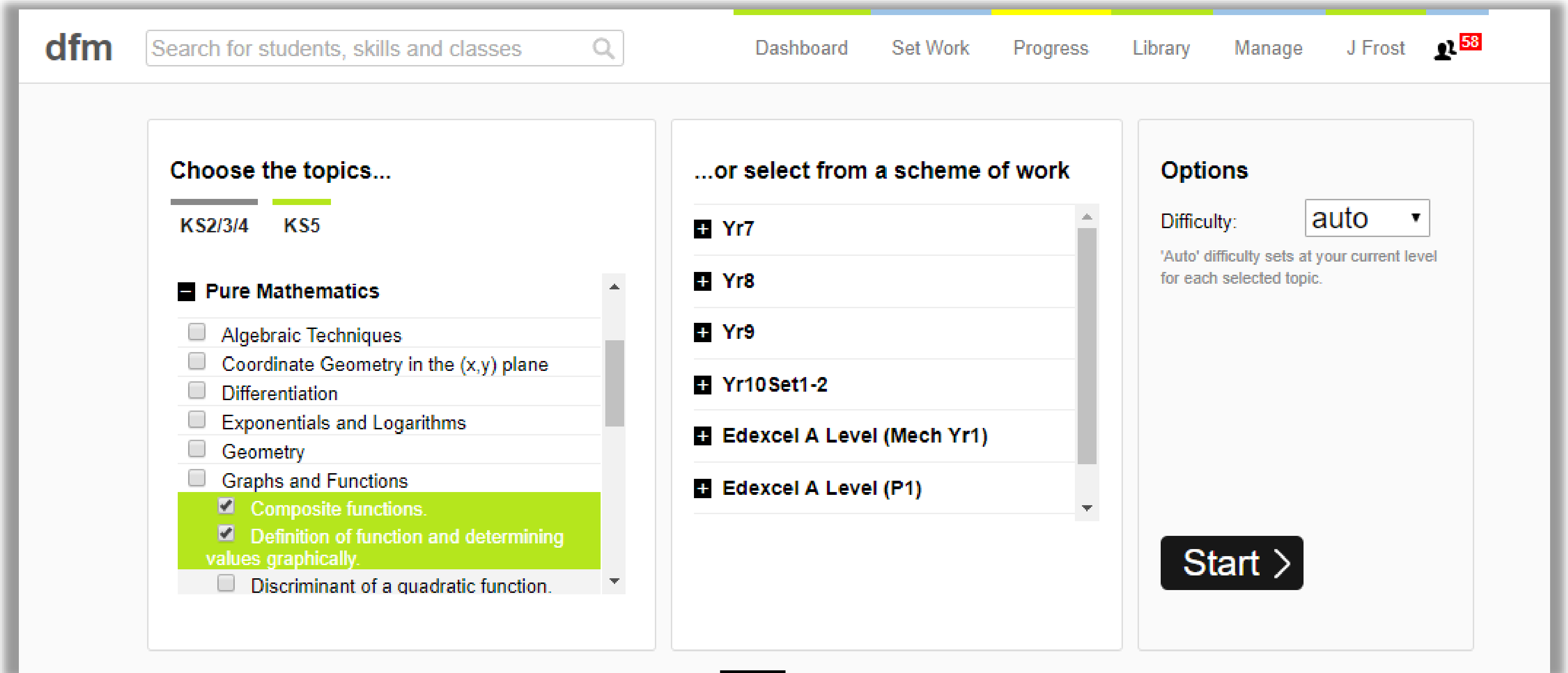
P2 Chapter 1 :: Algebraic Methods

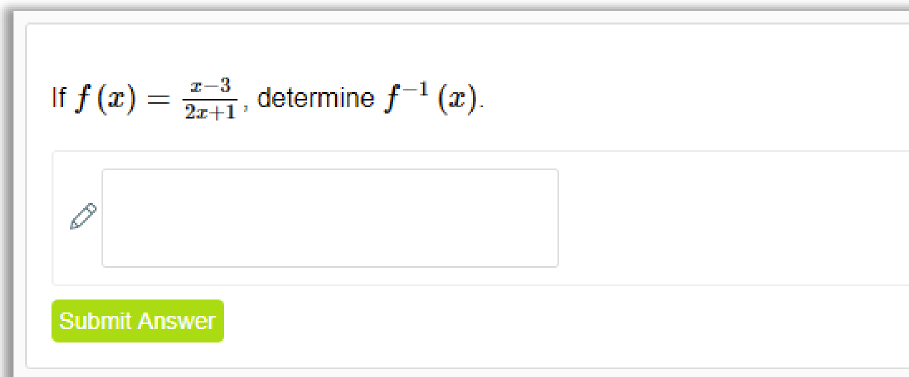
jfrost@tiffin.kingston.sch.uk

www.drfrostmaths.com
@DrFrostMaths

Last modified: 9th October 2017

Use of DrFrostMaths for practice





Register for **free** at:
www.drfrostmaths.com/homework

Practise questions by chapter, including past paper Edexcel questions and extension questions (e.g. MAT).

Teachers: you can create student accounts (or students can register themselves).

Chapter Overview

Apart from the initial portion on proof, this chapter concerns manipulation of algebraic fractions.

1:: Proof By Contradiction

Prove by contradiction that $\sqrt{5}$ is an irrational number.

2:: $+\div\times-$ Algebraic Fractions

[Edexcel C3] Express

$$\frac{3x+5}{x^2+x-12}-\frac{2}{x-3}$$

as a single fraction in its simplest form.

3:: Express a fraction using partial fractions.

Express $\frac{2+3x}{(x+3)(x-1)}$ in the form $\frac{A}{x+3}+\frac{B}{x-1}$ where A and B are constants.

4:: Divide algebraic expressions, and convert an improper fraction into partial fraction form.

Given that

$$\frac{x^3+x^2+7}{x-3}\equiv Ax^2+Bx+C+\frac{D}{x-3}$$

find the values of A, B, C and D.

1 :: Proof By Contradiction

✎ To prove a statement is true by contradiction:

- **Assume** that the statement is in fact **false**.
- Prove that this would **lead to a contradiction**.
- Therefore we were wrong in assuming the statement was false, and therefore it must be true.

Prove that there is no greatest odd integer.

? Assumption
? Show contradiction
? Conclusion

How to structure/word proof:

1. "Assume that *[negation of statement]*."
2. [*Reasoning followed by…*] *"This contradicts the assumption that…"* or *"This is a contradiction"*.
3. "Therefore [*restate original statement*]."

Negating the original statement

The first part of a proof by contradiction requires you to negate the original statement. What is the negation of each of these statements? ***(Click to choose)***

"There are infinitely many prime numbers."

"All Popes are Catholic."

"If it is raining, my garden is wet."

"There are infinitely many non-prime (i.e. composite) numbers."

"There are finitely many prime numbers."

"There are finitely many non-composite numbers."

"There exists a Pope who is not Catholic."

"No Popes are Catholic."

"Dr Frost is the Pope."

"If it is not raining, my garden is dry."

"If it is not raining, my garden is wet."

"If it is raining, my garden is not wet."

Comments: The negation of "all are" is not "none are". So the negation of "everyone likes green" wouldn't be "no one likes green", but: "not everyone likes green". Do not confuse a 'negation' with the 'opposite'.

Comments: If you have a conditional statement like "*If A then B*", then the negation is "*If A then not B*", i.e. the same condition applies, but the implication is negated.

More Examples

Prove by contradiction that if n^2 is even, then n must be even.

? Assumption

? Show contradiction

? Conclusion

Remember the negation of "if A then B" is "if A, then not B".

More Examples

Prove by contradiction that $\sqrt{2}$ is an irrational number.

? Assumption

? Show contradiction

? Conclusion

A **rational number** is one that can be expressed in the form $\frac{a}{b}$ where a, b are integers.

An **irrational** number cannot be expressed in this form, e.g. $\pi, e, \sqrt{3}$.

The set of all rational numbers is $\mathbb{Q}$ (real numbers: $\mathbb{R}$, natural numbers: $\mathbb{N}$, integers: $\mathbb{Z}$).

This is the standard (and well known) proof for the irrationality of $\sqrt{2}$. Here's Dr Frost's non-standard but quicker proof:

$$2b^2 = a^2$$

If a number is square then the powers in the prime factorisation are even. The power of 2 on the RHS is therefore even, but odd on the LHS (due to the extra 2). This is a contradiction.

More Examples

Prove by contradiction that there are infinitely many prime numbers.

? Assumption

? Show contradiction

? Conclusion

This proof is courtesy of Euclid, and is one of the earliest known proofs.

Exercise 1A

Pearson Pure Mathematics Year 2/AS
Pages 4-5

2a :: Multiplying/Dividing Algebraic Fractions

As your saw at GCSE level, multiplying algebraic fractions is no different to multiply numeric fractions.
You may however need to **cancel common factors, by factorising where possible**.

$$\frac{a}{b} \times \frac{c}{a} = \boxed{?}$$

$$\frac{x+1}{2} \times \frac{3}{x^2-1} = \boxed{?}$$

To divide by a fraction, **multiply by the reciprocal of the second fraction**.

$$\frac{p}{q} \div \frac{r}{q} = \boxed{?}$$

$$\frac{x+2}{x+4} \div \frac{3x+6}{x^2-16} = \boxed{?}$$

Test Your Understanding

$$\frac{x+3}{5} \times \frac{10}{x^2-9} =$$

?

$$\frac{x^2+x}{y} \div \frac{x^2-x-2}{y^2} =$$

?

Common student "Crime against Mathematics":

$$\frac{x^2+\cancel{y}}{2\cancel{y}} = \frac{x^2}{2}$$

?

Exercise 1B

Pearson Pure Mathematics Year 2/AS
Pages 6-7

2b :: Adding/Subtracting Algebraic Fractions

To add/subtract two fractions, find a common denominator.

This can be achieved by multiplying the denominators and cross-multiplying the numerators:

$$\frac{3}{x+1} - \frac{2}{x+2} =$$

?

However, often we should see if first if there are common factors in the denominator **to avoid multiplying unnecessarily**:

$$\frac{3}{x+1} - \frac{4x}{x^2-1}$$

?

Test Your Understanding

[Edexcel C3 June 2013(R) Q1]

Express

$$\frac{3x+5}{x^2+x-12} - \frac{2}{x-3}$$

as a single fraction in its simplest form.

?

Express the following as a single fraction, giving your answer in its simplest form.

$$\frac{10x+4}{3x^2+4x+1} - \frac{3}{x+1}$$

?

Exercise 1C

Pearson Pure Mathematics Year 2/AS
Page 8

Partial Fractions

We often want to reverse the process of combining fractions to split a fraction up into its components or partial fractions. We will consider 4 scenarios which depend on the nature of the denominator.

1. Two distinct linear factors $$\frac{6x-2}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$$

2. Three distinct linear factors $$\frac{6x^2+5x-2}{x(x-1)(2x+1)} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{2x+1}$$

3. A repeated linear factor $$\frac{11x^2+14x+5}{(x+1)^2(2x+1)}$$

4. Improper Fractions $$\frac{x^2+5x-9}{x+2}$$

Partial Fractions 1: Two linear factors

If the **denominator is a product of a linear terms**, it can be split into the sum of 'partial fractions', where **each denominator is a single linear term**. We can use either one of 2 methods to find the values of A and B.

$$\frac{6x-2}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$$

Notation reminder: $\equiv$ means 'equivalent/identical to', and indicates that both sides are equal for <u>**all**</u> values of x.

Method 1: Substitution

?.

Method 2: Comparing Coefficients

?

Partial Fractions 1 : Two linear factors

Worked Example: Express the following fraction in partial fractions.

$$\frac{2x+1}{(x+2)(x-1)}$$

?

Test Your Understanding

C4 June 2005 Q3a

Express $\dfrac{5x+3}{(2x-3)(x+2)}$ in partial fractions.

(3)

?

Partial Fractions 2: Three distinct linear factors

We use the same technique for 3 distinct linear factors but sensible substitution is the preferred method for solving as comparing coefficients can become complicated.

Given that $\frac{6x^2+5x-2}{x(x-1)(2x+1)} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{2x+1}$, find the values of the constants A, B, C.

?

Test Your Understanding

Express in partial fractions $$\frac{-5x^2-19x-32}{(x+1)(x+2)(x-5)}$$

?

Exercise 1D

Pearson Pure Mathematics Year 2/AS
Page 11

Partial Fractions 3: Repeated linear factors

Suppose we wished to express $\frac{2x+1}{(x+1)^2}$ as $\frac{A}{x+1}+\frac{B}{x+1}$. What's the problem?

?

Q Split $\frac{11x^2+14x+5}{(x+1)^2(2x+1)}$ into partial fractions.

$$\frac{11x^2+14x+5}{(x+1)^2(2x+1)} \equiv \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{2x+1}$$

The problem is resolved by having the factor **both squared and non-squared**.

?

Test Your Understanding

C4 June 2011 Q1

$$\frac{9x^2}{(x-1)^2(2x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(2x+1)}.$$

Find the values of the constants A, B and C. **(4)**

?

Exercise 1E

Pearson Pure Mathematics Year 2/AS

Page 13

Partial Fractions 4: Dealing with Improper Fractions

In Pure Year 1, we saw that the '**degree**' of a polynomial is the highest power, e.g. a quadratic has degree 2.

An algebraic fraction is **improper** if the degree of the numerator is **at least** the degree of the denominator.

$$\frac{x^2 - 3}{x + 2} \qquad \frac{x + 1}{x - 1} \qquad \frac{x^3 - x^2 + 3}{x^2 - x}$$

A partial fraction is still improper if the degree is the same top and bottom.

Questions might take one of two forms:

- Do the division to express as a quotient and a remainder, e.g. $\frac{x+1}{x-1} \rightarrow 1 + \frac{2}{x-1}$
- Express as partial fractions, e.g. $\frac{x^2+x}{(x+1)(x-2)} = A + \frac{B}{x+1} + \frac{C}{x-2}$

Reducing to Quotient and Remainder

You know for example that as $7 \div 3 = 2\ rem\ 1$, we could write:

$$\frac{7}{3} = 2 + \frac{1}{3}$$

Similarly in general:

$$\frac{F(x)}{divisor} = \underbrace{Q(x)}_{\text{Quotient}} + \frac{remainder}{divisor}$$

If $\frac{x^2+5x-9}{x+2} = Ax + B + \frac{C}{x+2}$, determine the values of A, B and C.

?

Test Your Understanding

Edexcel C4 June 2013 Q1

Fro Tip: There's a missing x term in the numerator and missing x term in the denominator. Use $+0x$ to avoid gaps.

Given that

$$\frac{3x^4 - 2x^3 - 5x^2 - 4}{x^2 - 4} \equiv ax^2 + bx + c + \frac{dx + e}{x^2 - 4}, \quad x \neq \pm 2$$

find the values of the constants a, b, c, d and e.

(4)

?

Exercise 1F

Pearson Pure Mathematics Year 2/AS
Pages 16-17

Dealing with Improper Fractions

Q Split $\frac{3x^2-3x-2}{(x-1)(x-2)}$ into partial fractions.

Method 1: Algebraic Division

?

Method 2: Using One Identity

?

Test Your Understanding

C4 Jan 2013 Q3

Express $\dfrac{9x^2+20x-10}{(x+2)(3x-1)}$ in partial fractions.

(4)

?

Exercise 1G

Pearson Pure Mathematics Year 2/AS
Page 18