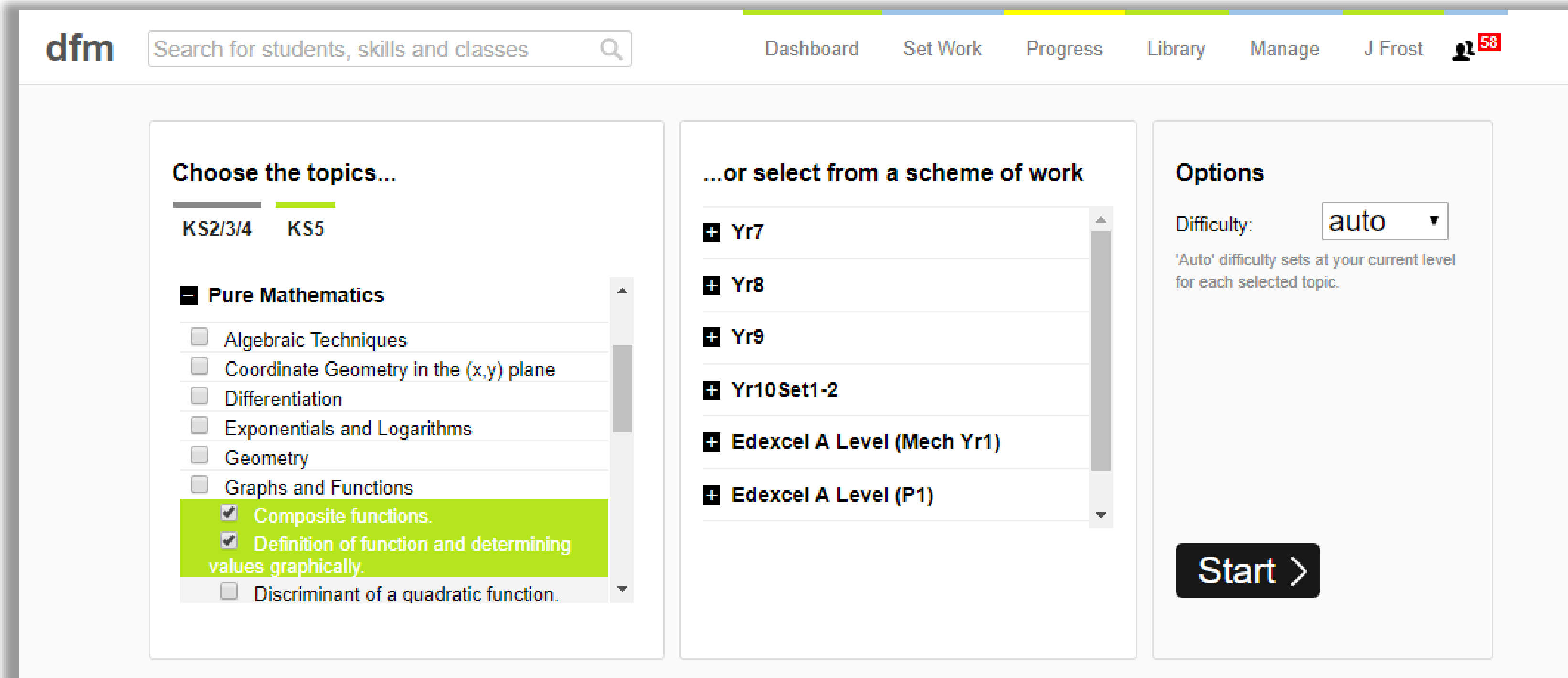
P2 Chapter 5 :: Radians

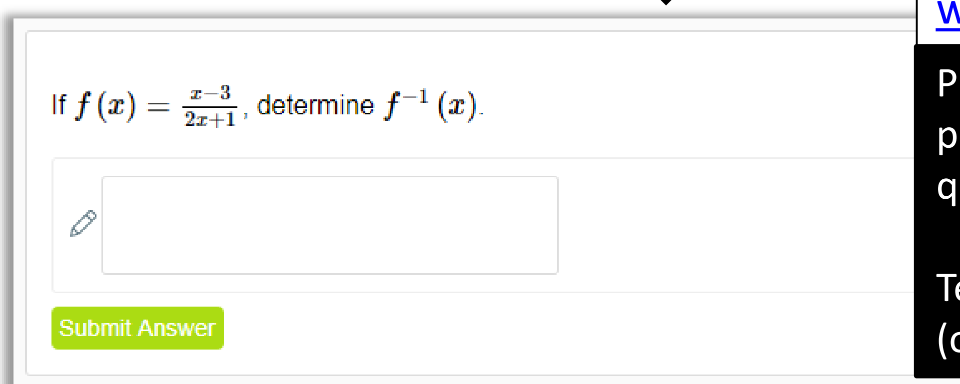
jfrost@tiffin.kingston.sch.uk

www.drfrostmaths.com
@DrFrostMaths

Last modified: 8th November 2017

Use of DrFrostMaths for practice





Register for **free** at:
www.drfrostmaths.com/homework

Practise questions by chapter, including past paper Edexcel questions and extension questions (e.g. MAT).

Teachers: you can create student accounts (or students can register themselves).

Chapter Overview

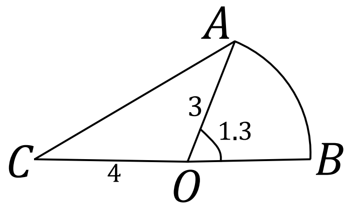
The concept of radians will likely be completely new to you.

1:: Converting between degrees and radians.

"What is $45°$ in radians?"

2:: Find arc length and sector area (when using radians)

"OAB is a sector. Determine the perimeter of the shape."



3:: Solve trig equations in radians.

"Solve $\sin x = \frac{1}{2}$ for $0 \leq x < \pi$."

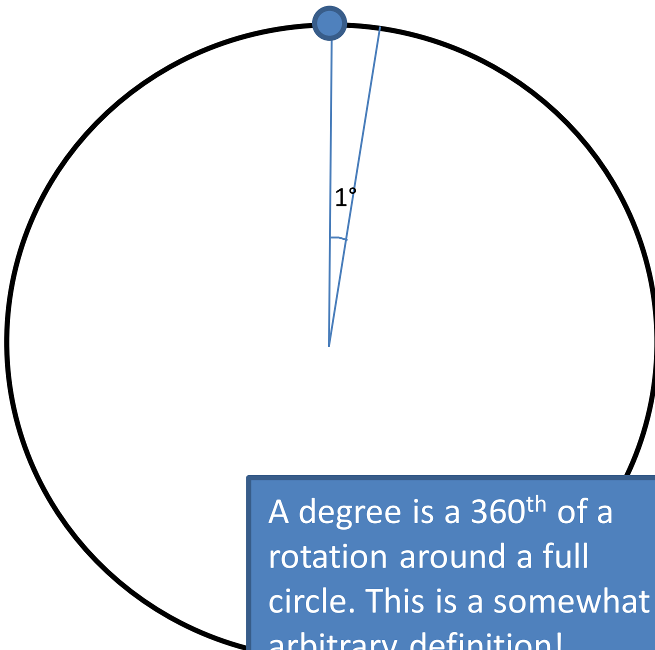
4:: Small angle approximations

"Show that, when θ is small, $\sin 5\theta + \tan 2\theta - cos2\theta \approx 2\theta^2 + 7\theta - 1$."

***NEW* to A Level 2017!**

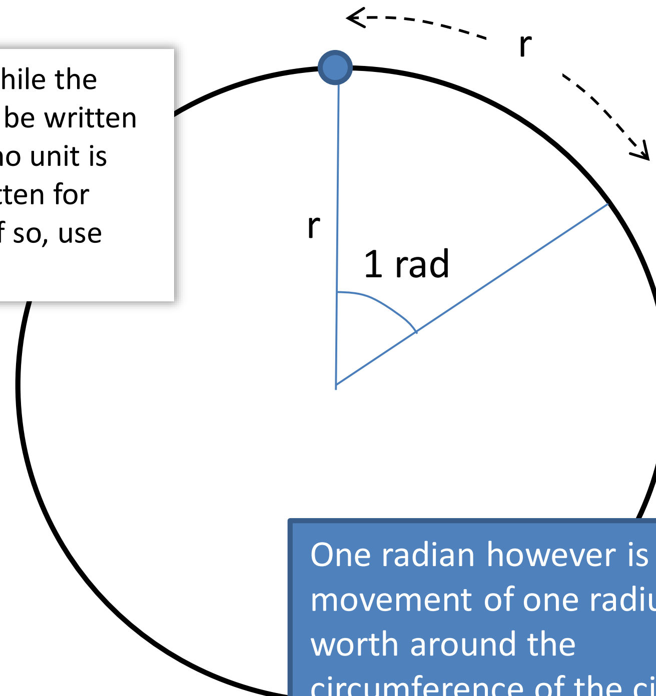
Radians

So far you've used **degrees** as the unit to measure angles.
But outside geometry, mathematicians pretty much always use **radians**.



A degree is a 360^{th} of a rotation around a full circle. This is a somewhat arbitrary definition!

Unit Note: While the unit "°" must be written for degrees, no unit is generally written for radians (but if so, use "rad").



One radian however is the movement of one radius' worth around the circumference of the circle.

Click to Start Degree animation

Click to Start Radian animation

Thinking about how many radii around the circumference we can go: ?

…but **WHY** use radians as our unit?

In later chapters we will see that $\sin x$ differentiates to $\cos x$ for example, but **ONLY** if x is in radians (and we will prove why).

Since trigonometric functions are commonly used in calculus (differentiation and integration) and that calculus underpins so many branches of maths, it explains why radians is seen as the 'default' unit for angles.

$$\frac{d}{dx}(\sin x) = \cos x$$

Converting between radians and degrees

?

$$180^\circ = \pi$$

?

$90^\circ =$?

$\dfrac{\pi}{3} =$?

$45^\circ =$?

$\dfrac{\pi}{6} =$?

$135^\circ =$?

$\dfrac{3}{2}\pi =$?

$72^\circ =$?

$\dfrac{5\pi}{6} =$?

Be able to convert common angles in your head...

$45° = \boxed{?}$

$30° = \boxed{?}$

$60° = \boxed{?}$

$135° = \boxed{?}$

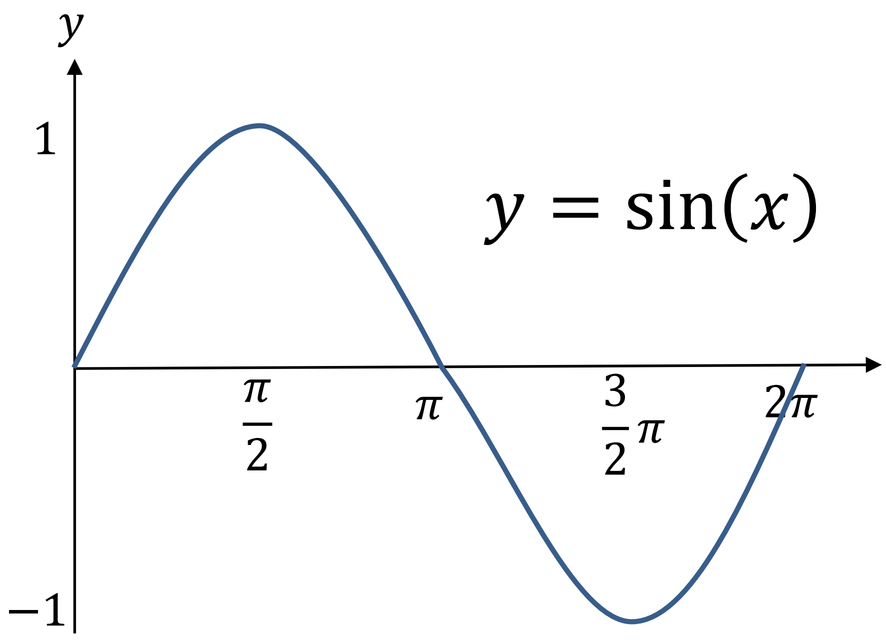
$270° = \boxed{?}$

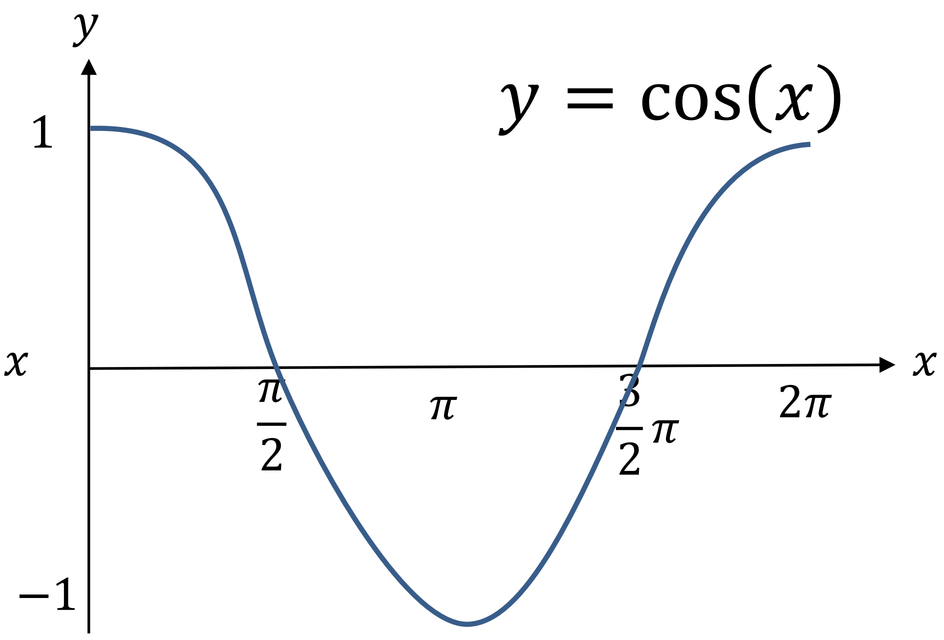
$90° = \boxed{?}$

$120° = \boxed{?}$

Graph Sketching with Radians

We can replace the values $90°, 180°, 270°, 360°$ on the x-axis with their equivalent value in radians.





Test Your Understanding

Sketch the graph of $y = \cos\left(x + \frac{\pi}{2}\right)$ for $0 \leq x < 2\pi$.

?

sin, cos, tan of angles in radians

Reminder of laws from Year 1:

- $\sin(x) = \sin(180 - x)$
- $\cos(x) = \cos(360 - x)$
- sin, cos repeat every $360°$ but tan every $180°$

- $\sin(x) = \sin(\pi - x)$
- $\cos(x) = \cos(2\pi - x)$
- sin, cos repeat every 2π but tan every π

To find sin/cos/tan of a ‘**common**’ angle in radians without using a calculator, it is easiest to just **convert to degrees first**.

$\cos\left(\frac{4\pi}{3}\right) =$?

$\sin\left(-\frac{7\pi}{6}\right) =$?

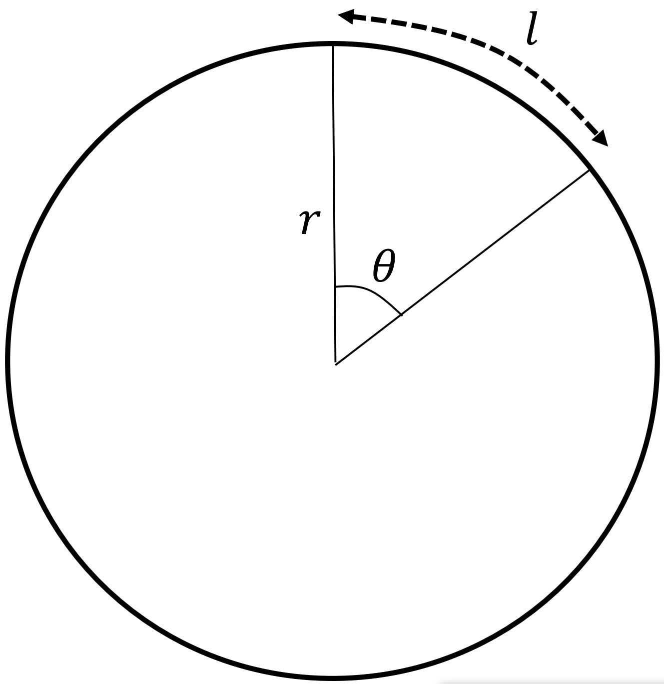
“Use of Technology” Monkey says:

To find cos $\left(\frac{4\pi}{3}\right)$ directly using your calculator, you need to switch to radians mode. Press $SHIFT \rightarrow SETUP$, then $ANGLE\ UNIT$, then $Radians$. An R will appear at the top of your screen, instead of D.

Exercises 5A/5B

Pearson Pure Mathematics Year 2/AS
Pages 116, 118

Arc length



Arc length in degrees:

?

Arc length in radians

From before, we know that 1 radian gives an arc of 1 radius in length, so θ radians must give a length of…

?

Examples

[Textbook] Find the length of the arc of a circle of radius 5.2 cm, given that the arc subtends an angle of 0.8 radians at the centre of the circle.

?

[Textbook] An arc AB of a circle with radius 7 cm and centre O has a length of 2.45 cm. Find the angle $\angle AOB$ subtended by the arc at the centre of the circle

?

Fro Note: Whether your calculator is in degrees mode or radians mode is only relevant when using sin/cos/tan – it won't affect simple multiplication!

Terminology: '<u>Subtend</u>' means **opposite** or extending beneath.

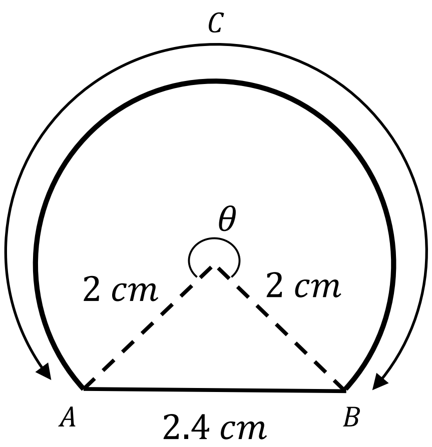
Further Examples

[Textbook] An arc AB of a circle, with centre O and radius r cm, subtends an angle of θ radians at O. The perimeter of the sector AOB is P cm. Express r in terms of P and θ.

?

[Textbook] The border of a garden pond consists of a straight edge AB of length 2.4m, and a curved part C, as shown in the diagram. The curve part is an arc of a circle, centre O and radius 2m.

Find the length of C.



?

Test Your Understanding

Edexcel C2 Jan 2005 Q7

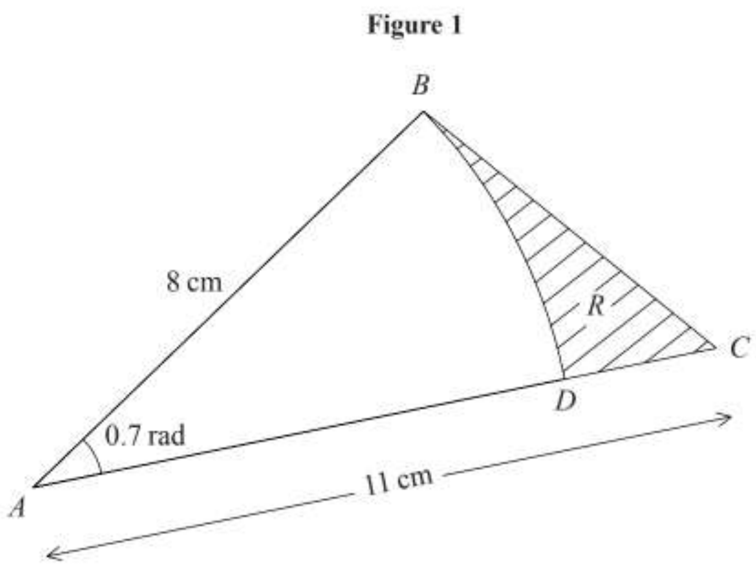


Figure 1 shows the triangle ABC, with $AB = 8\ cm$, $AC = 11\ cm$ and $\angle BAC = 0.7$ radians. The arc BD, where D lies on AC, is an arc of a circle with centre A and radius 8 cm. The region R, shown shaded in Figure 1, is bounded by the straight lines BC and CD and the arc BD.

Find

(a) The length of the arc BD.

(b) The perimeter of R, giving your answer to 3 significant figures.

a ?

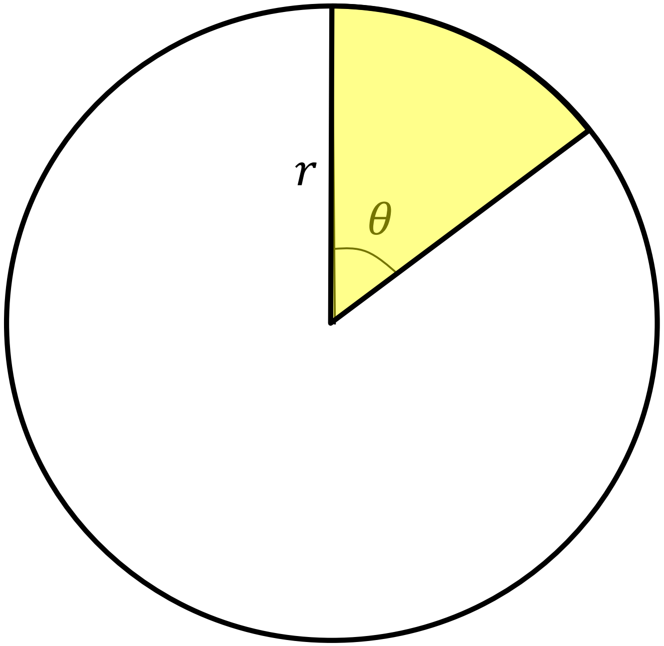
b ?

Exercises 5C

Pearson Pure Mathematics Year 2/AS
Pages 120-122

Note: Q10 is based on a past paper question so is worth doing.

Sector Area



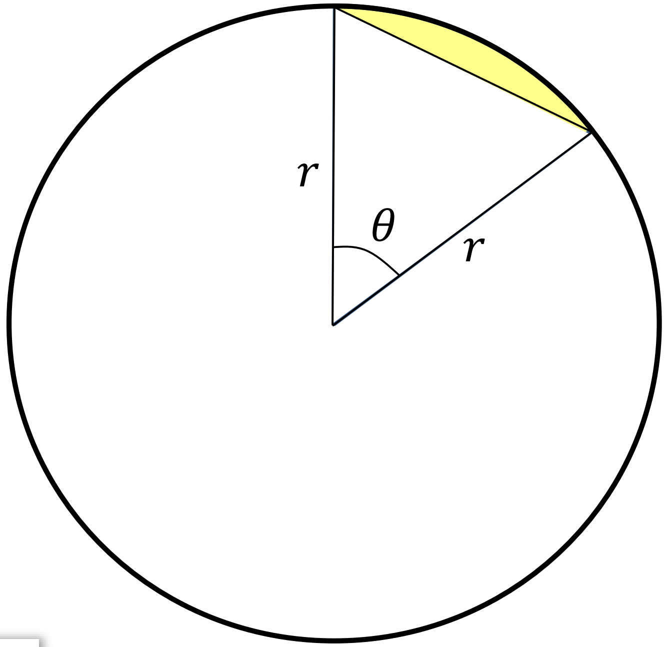
Area using Degrees

$A =$?

Area using Radians

$A =$?

Segment Area



A segment is the region bound between a chord and the circumference.

This is just a sector with a triangle cut out.

Area using radians:

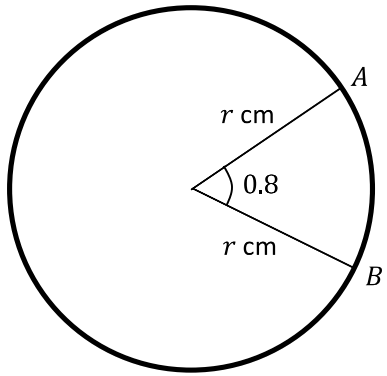
$A =$?

$=$

Recall that the area of a triangle is $\frac{1}{2}ab\sin C$ where C is the 'included angle' (i.e. between a and b)

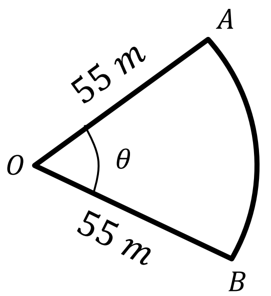
Examples

[Textbook] In the diagram, the area of the minor sector AOB is 28.9 cm^2. Given that $\angle AOB = 0.8$ radians, calculate the value of r.



?

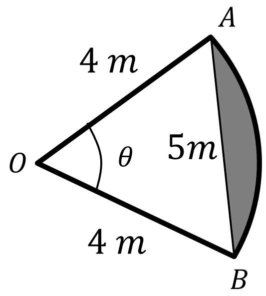
[Textbook] A plot of land is in the shape of a sector of a circle of radius 55 m. The length of fencing that is erected along the edge of the plot to enclose the land is 176 m. Calculate the area of the plot of land.



?

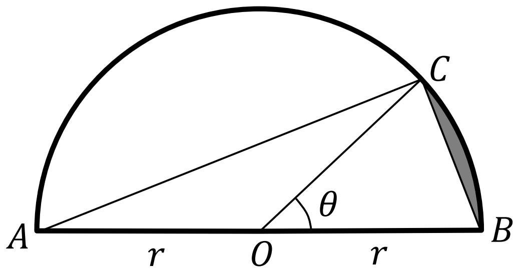
Segment Examples

[Textbook] In the diagram above, OAB is a sector of a circle, radius 4m. The chord AB is 5m long. Find the area of the shaded segment.



?

[Textbook] In the diagram, AB is the diameter of a circle of radius $r$cm, and $\angle BOC = \theta$ radians. Given that the area of ΔAOC is three times that of the shaded segment, show that
$3\theta - 4\sin\theta = 0$.



?

Test Your Understanding

Edexcel C2

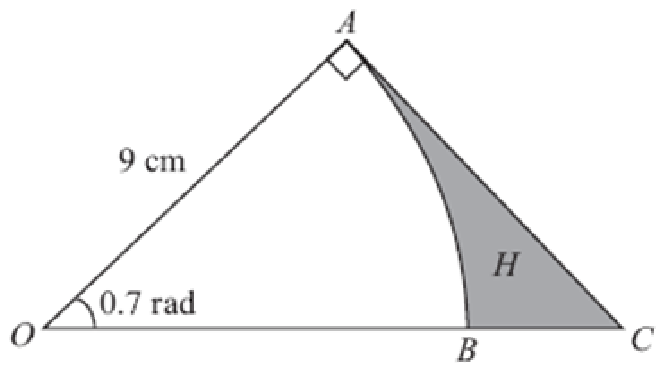


Figure 1

Figure 1 shows the sector OAB of a circle with centre O, radius 9 cm and angle 0.7 radians.

~~(a) Find the length of the arc AB.~~

(2)

(b) Find the area of the sector OAB.

?

(2)

The line AC shown in Figure 1 is perpendicular to OA, and OBC is a straight line.

?

(c) Find the length of AC, giving your answer to 2 decimal places.

(2)

The region H is bounded by the arc AB and the lines AC and CB.

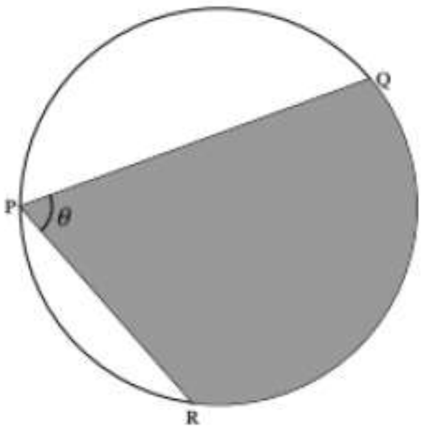
(d) Find the area of H, giving your answer to 2 decimal places.

?

Exercises 5D

Pearson Pure Mathematics Year 2/AS
Pages 125-128

Extension



[MAT 2012 1J]
If two chords QP and RP on a circle of radius 1 meet in an angle θ at P, for example as drawn in the diagram on the left, then find the largest possible area of the shaded region RPQ, giving your answer in terms of θ.

?

Solving Trigonometric Equations

- $\sin(x) = \sin(\pi - x)$
- $\cos(x) = \cos(2\pi - x)$
- sin, cos repeat every 2π but tan every π

Solving trigonometric equations is virtually the same as you did in Year 1, except:
(a) Your calculator needs to be in radians mode.
(b) We use $\pi -$ instead of $180° -$, and so on.

[Textbook] Solve the equation $\sin 3\theta = \frac{\sqrt{3}}{2}$ in the interval $0 \leq \theta \leq 2\pi$.

?

[Jan 07 Q6] Find all the solutions, in the interval $0 \leq x < 2\pi$, of the equation

$$2\cos^2 x + 1 = 5\sin x,$$

giving each solution in terms of π. **(6)**

?

Exercises 5E

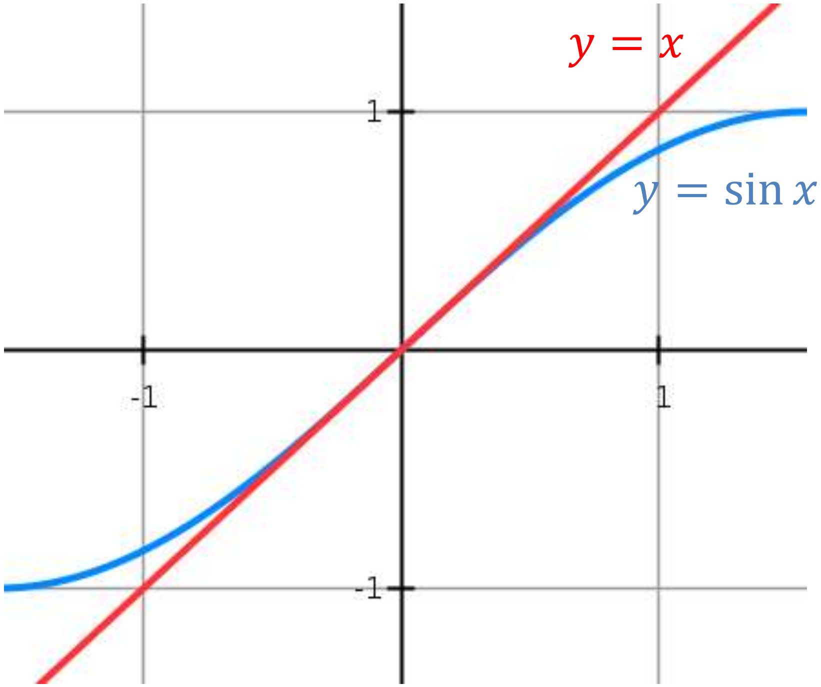
Pearson Pure Mathematics Year 2/AS
Pages 131-132

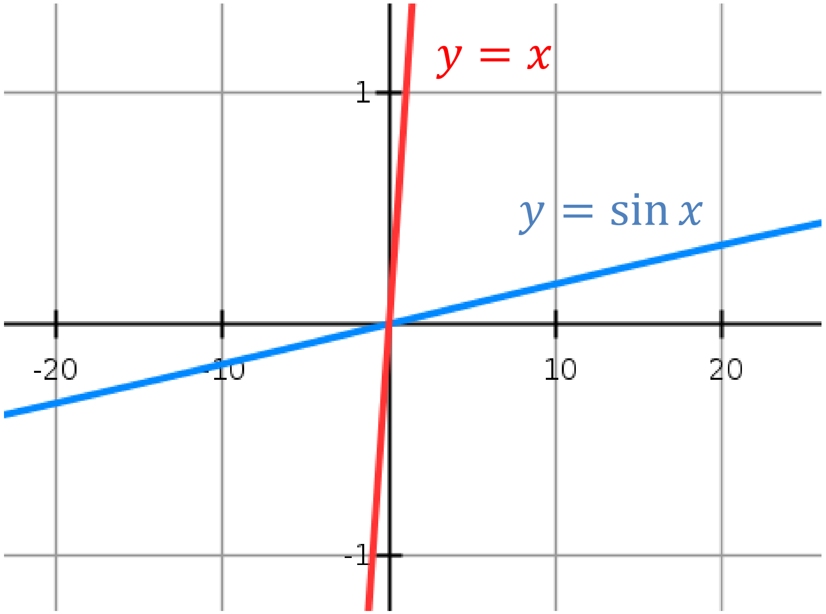
Extension

[MAT 2010 1C] In the range $0 \leq x \leq 2\pi$, the equation $\sin^2 x + 3\sin x \cos x + 2\cos^2 x = 0$ has how many solutions?

?

Small Angle Approximations





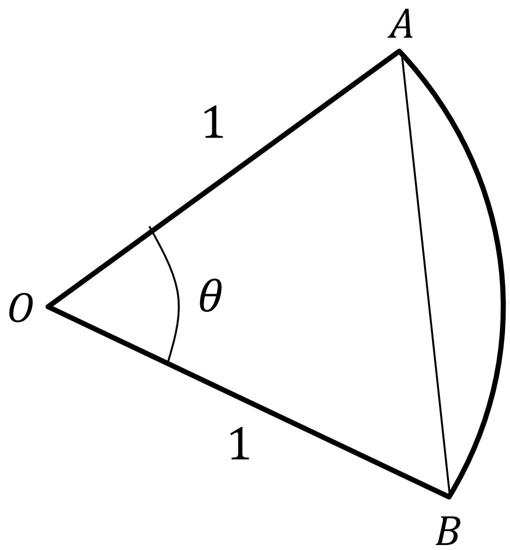
If x is in radians, we can see from the graph that as x approaches 0, the two graphs are approximately the same, i.e. $\sin x \approx x$

If x was in degrees however, then we can see this is not the case.

✏ When θ is small and measured in radians:

- $\sin\theta \approx \theta$
- $\tan\theta \approx \theta$
- $\cos\theta \approx 1 - \frac{\theta^2}{2}$

Small Angle Approximations



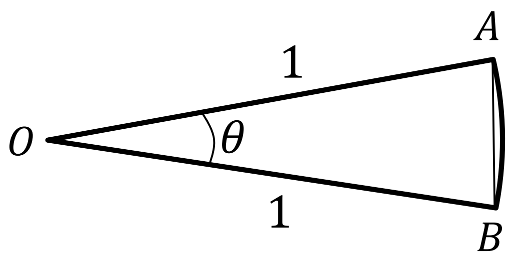
Geometric Proof that $\boldsymbol{sin\ \theta \approx \theta}$:

The area of sector OAB is:

$$\frac{1}{2} \times 1^2 \times \theta = \frac{1}{2}\theta$$

The area of triangle OAB is:

$$\frac{1}{2} \times 1^2 \times sin\theta = \frac{1}{2}\sin\theta$$



As θ becomes small, the area of the triangle is approximately equal to that of the sector, so:

$$\frac{1}{2}\sin\theta \approx \frac{1}{2}\theta$$

$$\sin\theta \approx \theta$$

Note that this only works for radians, because we used the sector area formula for radians. The fact that $\sin\theta \approx \theta$ is enormously important when we come to differentiation, because we can use it to prove that $\frac{d}{dx}(\sin x) = \cos x$.

Examples

When θ is small and measured in radians:
- $\sin\theta \approx \theta$
- $\tan\theta \approx \theta$
- $\cos\theta \approx 1 - \frac{\theta^2}{2}$

[Textbook] When θ is small, find the approximate value of:

a) $\frac{\sin 2\theta + \tan\theta}{2\theta}$

b) $\frac{\cos 4\theta - 1}{\theta \sin 2\theta}$

[Textbook] a) Show that, when θ is small,

$$\sin 5\theta + \tan 2\theta - \cos 2\theta \approx 2\theta^2 + 7\theta - 1$$

b) Hence state the approximate value of $\sin 5\theta + \tan 2\theta - \cos 2\theta$ for small values of θ.

a) ?

b) ?

?

?

Exercises 5F

Pearson Pure Mathematics Year 2/AS
Page 134