P1 Chapter 14 :: Exponentials & Logarithms

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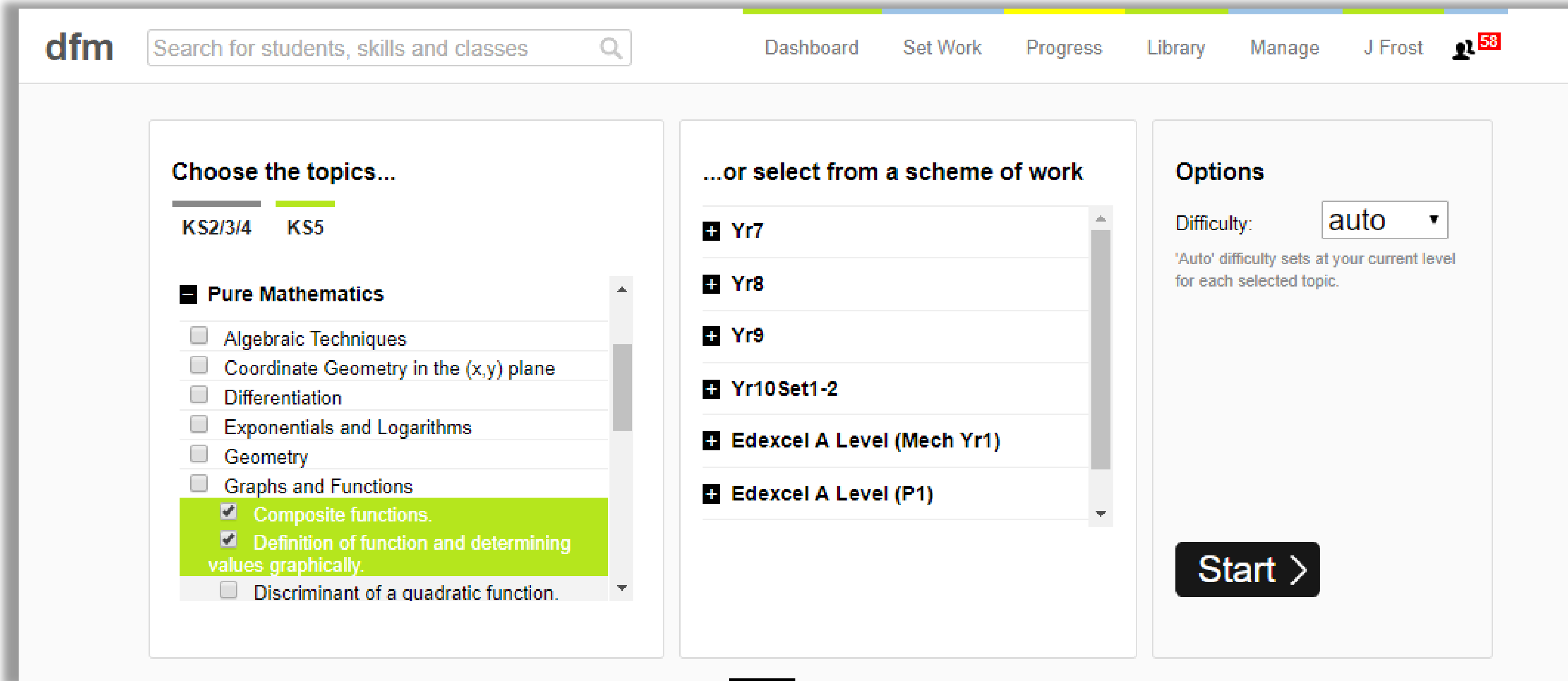

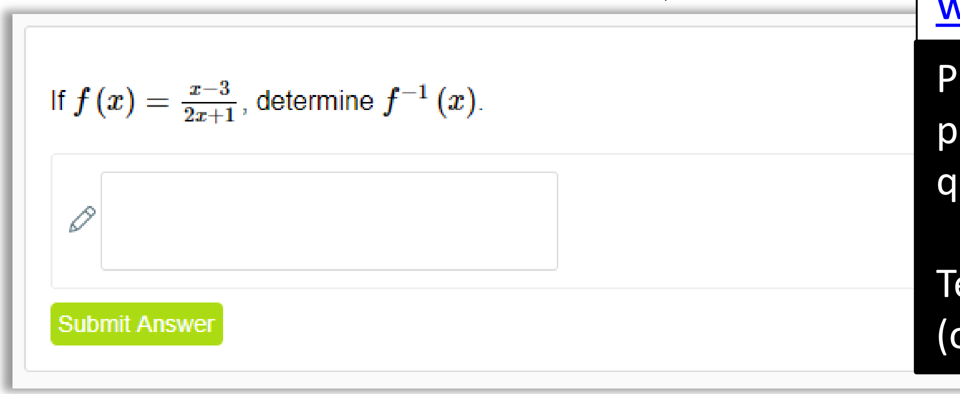

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Chapter Overview

You have encountered exponential expressions like $y = 2^x$ before, but probably not 'the' exponential function $y = e^x$. Similarly, you will learn that the inverse of $y = 2^x$ is $y = \log_2 x$.

1:: Sketch exponential graphs.

Sketch $y = 2^x$ and $y = e^x$ on the same axes.

NEW! to A Level 2017
"The" exponential function, e^x, has been moved from Year 2 to Year 1.

2:: Use an interpret models that use exponential functions.

The population P of Davetown after t years is modelled using $P = Ae^{kt}$, where A, k are constants…

NEW! to A Level 2017
Again, moved from Year 2.

3:: Be able to differentiate e^{kx}.

If $y = 5e^{3x}$, determine $\frac{dy}{dx}$.

4:: Understand the log function and use laws of logs.

Solve the equation:

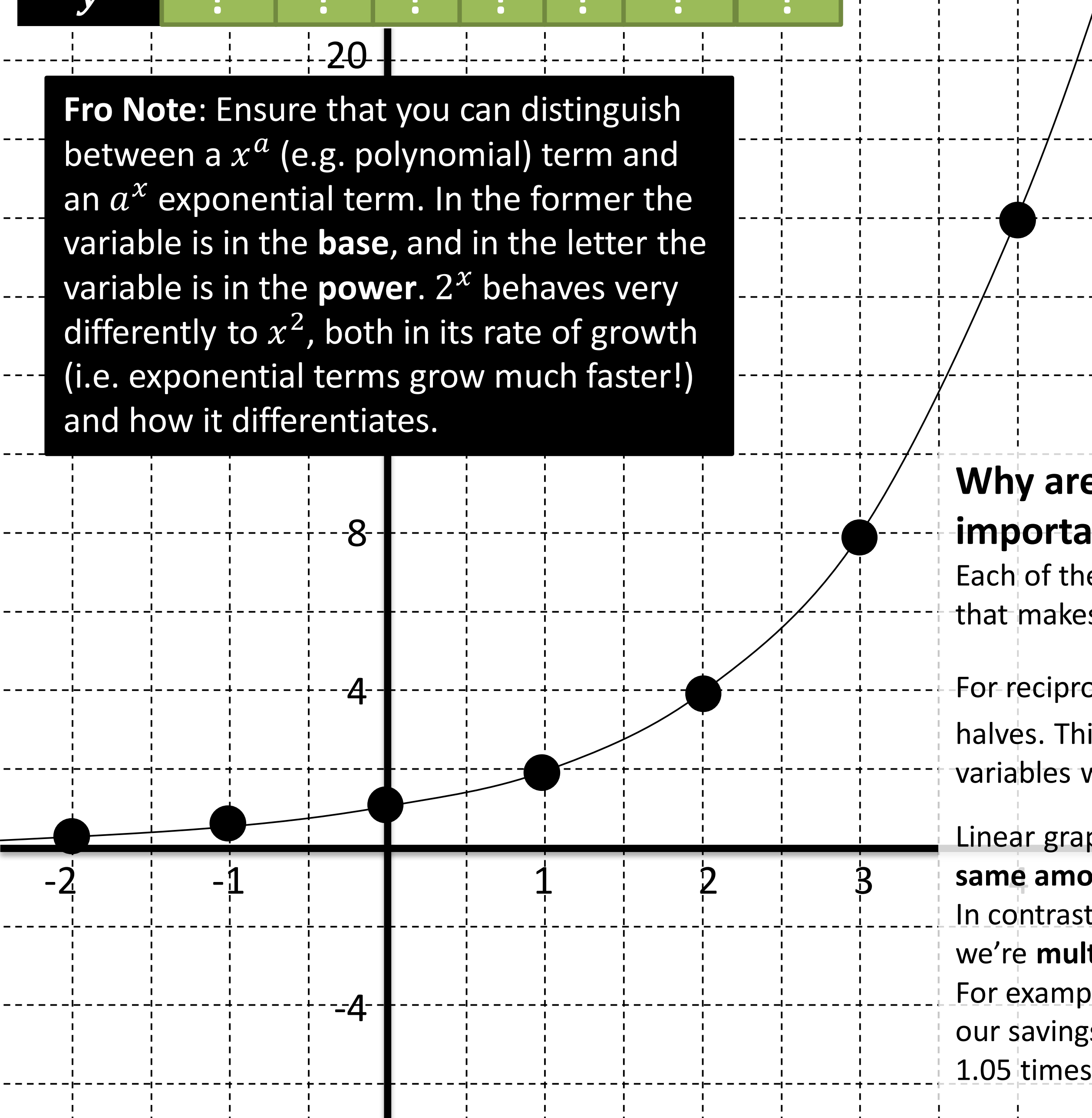
$$\log_2(2x) = \log_2(5x + 4) - 3$$

5:: Use logarithms to estimate values of constants in non-linear models.

(This is a continuation of (2))

x	-2	-1	0	1	2	3	4
y	?	?	?	?	?	?	?

$$y = 2^x$$



Fro Note: Ensure that you can distinguish between a x^a (e.g. polynomial) term and an a^x exponential term. In the former the variable is in the **base**, and in the letter the variable is in the **power**. 2^x behaves very differently to x^2, both in its rate of growth (i.e. exponential terms grow much faster!) and how it differentiates.

Why are exponential functions important?

Each of the common graphs have a **key property** that makes them **useful for modelling**.

For reciprocal graphs $y = \frac{k}{x}$, as x doubles, y halves. This means we'd use it to represent variables which are inversely proportional.

Linear graphs are used when we're **adding the same amount** each time.

In contrast, exponential graphs are used when we're **multiplying by the same amount** each time.
For example, we might use $1000(1.05^t)$ to model our savings with interest, where each year we have 1.05 times as much, i.e. with 5% added interest.

Contrasting exponential graphs

On the same axes sketch $y = 3^x, y = 2^x, y = 1.5^x$

?

On the same axes sketch $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$

?

Graph Transformations

Sketch $y = 2^{x+3}$

?

Exercise 14A

Pearson Pure Mathematics Year 1/AS

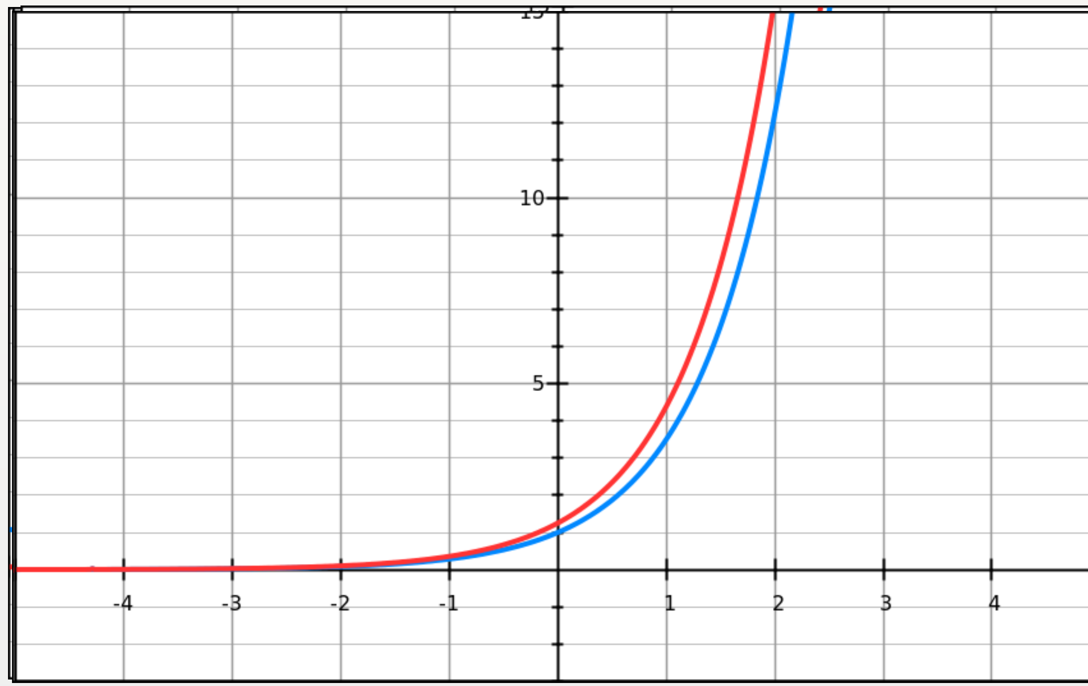
Pages 313-314

$y = e^x$

Click

Function	Gradient	
$y = 1^x$	$\frac{dy}{dx} = 0$	>
$y = 1.5^x$	$\frac{dy}{dx} = 0.41 \times 1.5^x$	>
$y = 2^x$	$\frac{dy}{dx} = 0.69 \times 2^x$	>
$y = 2.5^x$	$\frac{dy}{dx} = 0.92 \times 2.5^x$	>
$y = 3^x$	$\frac{dy}{dx} = 1.10 \times 3^x$	>
$y = 3.5^x$	$\frac{dy}{dx} = 1.25 \times 3.5^x$	>

Compare each exponential function against its respective gradient function. What do you notice?



$y = e^x$

Function	Gradient
$y = 1^x$	$\frac{dy}{dx} = 0$
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$y = 3^x$	$\frac{dy}{dx} = 1.10 \times 3^x$
$y = 3.5^x$	$\frac{dy}{dx} = 1.25 \times 3.5^x$

$y = 2.5^x$ and $y = 3^x$ seem to be similar to their respective gradient functions. So is there a base between 2.5 and 3 where the **function is equal to its gradient function**?

$e = 2.71828 \ldots$ is known as **Euler's Number.**

It is one of the five most fundamental constants in mathematics (0, 1, i, e, π).

It has the property that:

$$y = e^x \quad \rightarrow \quad \frac{dy}{dx} = e^x$$

Although any function of the form $y = a^x$ is known as **an** exponential function, e^x is known as "**the**" exponential function.

You can find the exponential function on your calculator, to the right (above the "ln" key)

Differentiating $y = ae^{kx}$

✎ If $y = e^{kx}$, where k is a constant, then $\frac{dy}{dx} = ke^{kx}$

Note: This is not a standalone rule but an application of something called the '*chain rule*', which you will encounter in Year 2.

Different e^{5x} with respect to x.

?

Different e^{-x} with respect to x.

?

Different $4e^{3x}$ with respect to x.

?

Note: In general, when you scale the function, you scale the derivative/integral.

More Graph Transformations

Sketch $y = e^{3x}$

?

Sketch $y = 5e^{-x}$

?

Sketch $y = 2 + e^{\frac{1}{3}x}$

?

Test Your Understanding

Sketch $y = e^{-2x} - 1$

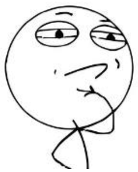
?

Exercise 14B

Pearson Pure Mathematics Year 1/AS
Pages 316-317

Just for your interest...

Where does e come from, and why is it so important?



$e = 2.71828\ldots$
is known as **Euler's Number**, and is considered one of the five fundamental constants in maths:
$0,\ 1,\ \pi,\ \ e,\ \ i$

Its value was originally encountered by Bernoulli who was solving the following problem:
You have £1. If you put it in a bank account with 100% interest, how much do you have a year later? If the interest is split into 2 instalments of 50% interest, how much will I have? What about 3 instalments of 33.3%? And so on…

No. Instalments	Money after a year
1	$1 \times 2^1 = £2$
2	$1 \times 1.5^2 = £2.25$
3	$1 \times 1.\dot{3}^3 = £2.37$
4	$1 \times 1.25^4 = £2.44$
n	$\left(1+\frac{1}{n}\right)^n$

As n becomes larger, the amount after a year approaches £2.71…, i.e. e!

Thus:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

But we have seen that differentiation by first principles uses 'limits'. It is therefore possible to prove from the definition above that $\frac{d}{dx}(e^x) = e^x$, and **these two definitions of *e* are considered to be equivalent***.

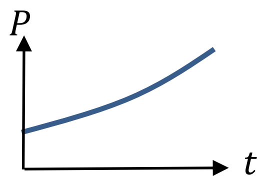
e therefore tends to arise in problems involving limits, and also therefore crops up all the time in anything involving differentiation and integration. Let's see some applications…

*You can find a full proof here in my Graph Sketching/Limits slides:
http://www.drfrostmaths.com/resources/resource.php?rid=163

Application 1: Solutions to many 'differential equations'.

Frequently in physics/maths, **the rate of change of a variable is proportional to the value itself**. So with a population P behaving in this way, if the population doubled, the rate of increase would double.

$$P \propto \frac{dP}{dt} \quad \rightarrow \quad P = k\frac{dP}{dt}$$



This is known as a 'differential equation' because the equation involves both the variable and its derivative $\frac{dP}{dt}$.

The 'solution' to a differentiation equation means to have an equation relating P and t without the $\frac{dP}{dt}$. We end up with (using Year 2 techniques):

$$\boldsymbol{P = Ae^{kt}}$$

where A and k are constants. This is expected, because we know from experience that population growth is usually exponential.

Application 2: Russian Roulette

A scene from one of Dr Frost's favourite films, *The Deer Hunter*.

I once wondered (as you do), if I was playing Russian Roulette, where you randomly rotate the barrel of a gun each time with n chambers, but with one bullet, what's the probability I'm still alive after n shots?

The probability of surviving each time is $1-\frac{1}{n}$, so the probability of surviving all n shots is $\left(1-\frac{1}{n}\right)^n$. We might consider what happens when n becomes large, i.e. $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$. In general, $e^k = \lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n$.

Thus $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1} = \frac{1}{e}$, i.e. I have a 1 in e chance of surviving. Bad odds!

This is also applicable to the lottery. If there was a 1 in 20 million chance of winning the lottery, we might naturally wonder what happens if we bought 20 million (random) lottery tickets. There's a 1 in e (roughly a third) chance of winning no money at all!

Application 3: Secret Santa

You might have encountered $n! = n \times (n-1) \times \cdots \times 2 \times 1$, said "$n$ factorial" meaning "*the number of ways of arranging n objects in a line*". So if we had 3 letters ABC, we have $3! = 6$ ways of arranging them.

ABC, ACB, BAC, BCA, CAB, CBA

Meanwhile, $!n$ means the number of **derangements** of n, i.e. the arrangements where **no letter appears in its original place**.

For ABC, that only gives BCA or CAB, so $!3 = 2$. This is applicable to '**Secret Santa**' (where each person is given a name out a hat of whom to give their present to) because ideally we want the scenario where *no person gets their own name*.

Remarkably, a derangement occurs an e-th of the time. So if there are 5 people and hence $5! = 120$ possible allocations of recipient names, we only get the ideal Secret Santa situation just $\frac{120}{e} = 44.15 \rightarrow 44$ times. And so we get **my favourite result in the whole of mathematics**:

$$!n = \left[\frac{n!}{e}\right]$$

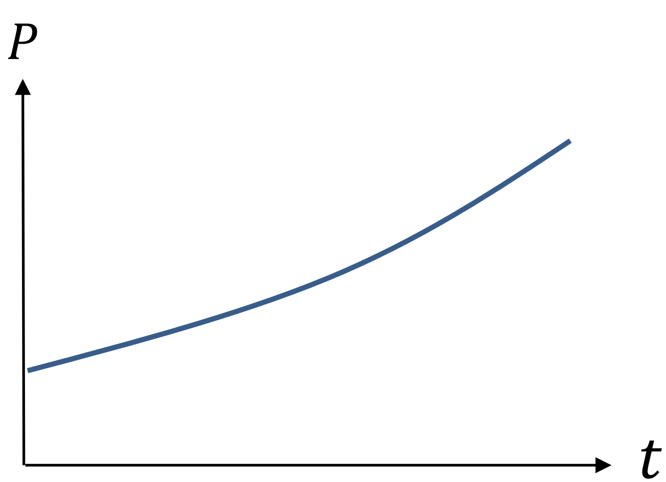
(where [...] means round)

Exponential Modelling

There are two key features of exponential functions which make them suitable for **population growth**:

1. **a^x gets a times bigger each time x increases by 1. (Because $a^{x+1} = a \times a^x$)**
With population growth, we typically have a fixed percentage increase each year. So suppose the growth was 10% a year, and we used the equivalent decimal multiplier, 1.1, as a. Then 1.1^t, where t is the number of years, would get 1.1 times bigger each year.
2. **The rate of increase is proportional to the size of the population at a given moment.**
This makes sense: The 10% increase of a population will be twice as large if the population itself is twice as large.

Suppose the population P in *The Republic of Dave*  is modelled by $P = 100e^{3t}$ where t is the numbers years since *The Republic* was established.



What is the initial population?

?

What is the initial rate of population growth?

?

"Use of Technology" Monkey says:
When I'm not busy eating ticks off other monkeys, I use the $e^{\square}$ key. I can also use [ALPHA] [e] to get e without a power.

Another Example

[Textbook] The density of a pesticide in a given section of field, P mg/m^2, can be modelled by the equation $P = 160e^{-0.006t}$
where t is the time in days since the pesticide was first applied.
a. Use this model to estimate the density of pesticide after 15 days.
b. Interpret the meaning of the value 160 in this model.
c. Show that $\frac{dP}{dt} = kP$, where k is a constant, and state the value of k.
d. Interpret the significance of the sign of your answer in part (c).
e. Sketch the graph of P against t.

a ?

In general, the 'initial value', when $t = 0$, is the coefficient of the exponential term.

b ?

c ?

d ?

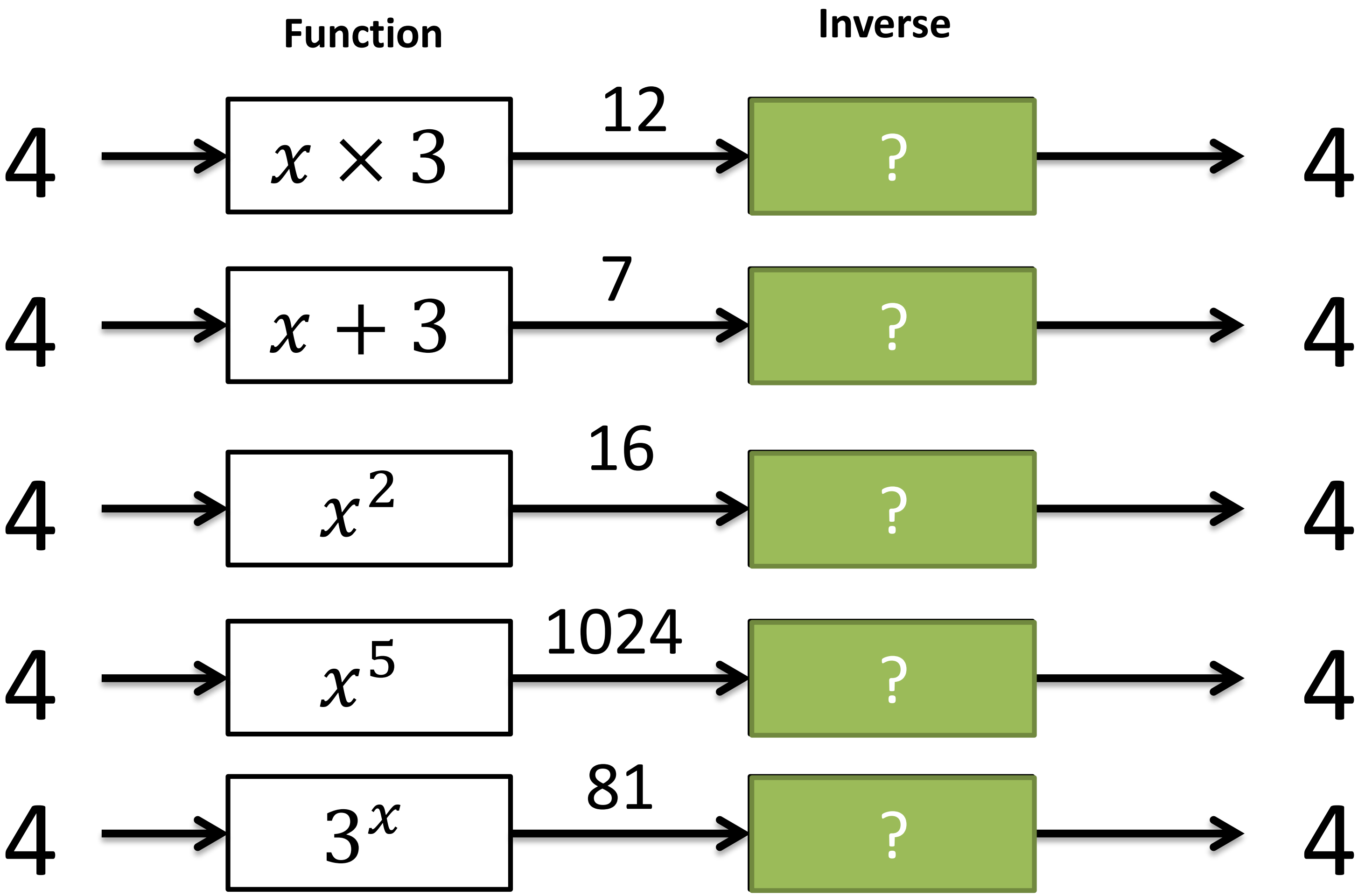
?

Exercise 14C

Pearson Pure Mathematics Year 1/AS
Pages 318-319

Logarithms

You know the inverse of many mathematical operations; we can undo an addition by 2 for example by subtracting 2. But is there an inverse function for an exponential function?



Such functions are known as **logarithms**, and exist in order to provide an inverse to exponential functions.

Interchanging between exponential and log form

$\log_a n$ ("said log base a of n") is equivalent to $a^x = n$.
The log function outputs the **missing power**.

$$3^2 = 9 \quad \Longleftrightarrow \quad \log_3 9 = 2$$

Here are two methods of interchanging between these forms.
Pick your favourite!

Method 1: 'Missing Power'

- Note first the base of the log must match the base of the exponential.
- $\log_2 8$ for example asks the question "2 to **what power** gives 8?"

We can imagine inserting the output of the log just after the base. Click the button!

$$log_2 \quad 8 = 3$$

Click to start Fro-manimation

Method 2: Do same operation to each side of equation.

Since KS3 you're used to the idea of doing the same thing to each side of the equation that 'undoes' whatever you want to get rid of.

$$3x + 2 = 11$$
$$(-2) \qquad (-2)$$
$$3x = 9$$

"log base a" undoes "a to the power of" and vice versa, as they are inverse functions.

$$\log_2 8 = 3$$
$$(2^{\square}) \qquad (2^{\square})$$
$$8 = 2^3$$

Examples

$$\log_5 25 = \boxed{?}$$

Think: "5 to the power of what gives you 25?"

$$\log_3 81 = \boxed{?}$$

$$\log_2 32 = \boxed{?}$$

$$\log_{10} 1000 = \boxed{?}$$

$$\log_4 1 = \boxed{?}$$

✎ $\log_a 1 = 0$ for all a.

$$\log_4 4 = \boxed{?}$$

$$\log_2 \left(\frac{1}{2}\right) = \boxed{?}$$

$$\log_3 \left(\frac{1}{27}\right) = \boxed{?}$$

$$\log_2 \left(\frac{1}{16}\right) = \boxed{?}$$

$$\log_a (a^3) = \boxed{?}$$

$$\log_4 (-1) = \boxed{?!!??}$$

While a log can output a negative number, we **can't log negative numbers**.

Strictly Just For Your Interest: However, if we were to expand the range (i.e. output) of the log function to allow *complex numbers* (known as the '*complex logarithm*'), then we in fact get $\log_4(-1) = \frac{i\pi}{\ln(4)}$. It's probably better if you purge these last few sentences from your memory and move along…

With Your Calculator

There are three buttons on your calculator for computing logs:

$\log_{\square} \square$

$\log_3 7 =$?

$\log_5 0.3 =$?

I couldn't think of a word that rhymed with 'ln' so I recorded it for you.

ln of x

ln

$\ln 10 =$?

$\ln e =$?

ln is the "**natural log of x**", meaning "log to the base e", i.e. it the inverse of e^x.

$$\ln(x) = \log_e(x)$$

We will use it more extensively later this chapter.

log

$\log 100 =$?

Just like the $\sqrt{}$ symbol without a number is $\sqrt[2]{\square}$ by default, log without a base is **base 10** by default when used on your calculator (although confusingly "log" can mean "ln" in mathematical papers)

Exercise 14D

Pearson Pure Mathematics Year 1/AS
Pages 320-321

Non-calculator!

Extension

1 [MAT 2015 1J] Which is the largest of the following numbers?

A) $\frac{\sqrt{7}}{2}$ B) $\frac{5}{4}$ C) $\frac{\sqrt{10!}}{3(6!)}$

D) $\frac{\log_2 30}{\log_3 85}$ E) $\frac{1+\sqrt{6}}{3}$

?

2 *[MAT 2013 1F]* Three *positive* numbers a, b, c satisfy

$$\log_b a = 2$$
$$\log_b(c-3) = 3$$
$$\log_a(c+5) = 2$$

This information:

A) specifies a uniquely;

B) is satisfied by two values of a;

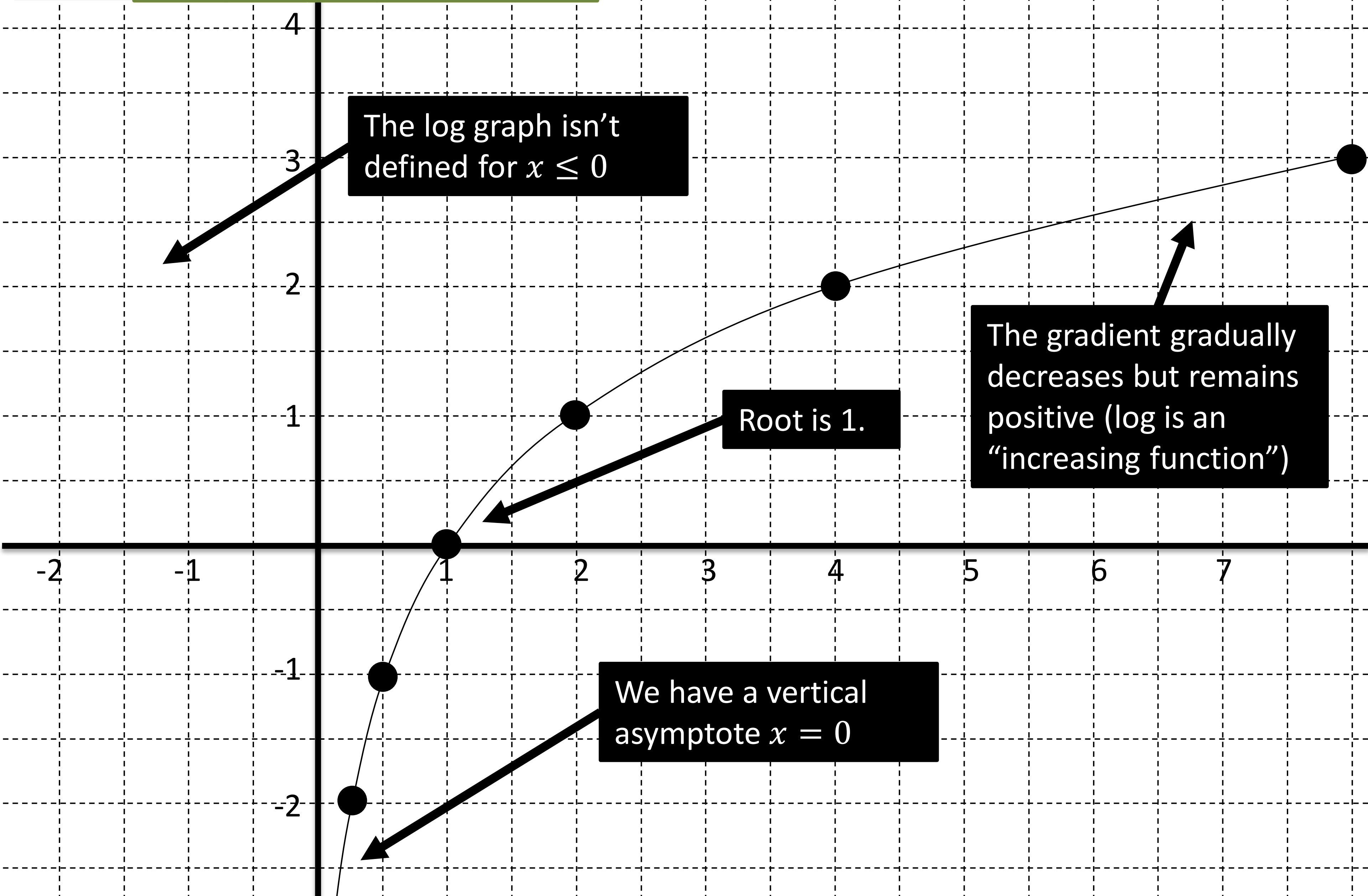
C) is satisfied by infinitely many values of a;

D) is contradictory

?

x	0.25	0.5	1	2	4	8
y	?	?	?	?	?	?

$$y = \log_2 x$$



Laws of Logs

Three main laws:

$$\log_a x + \log_a y = \log_a xy$$

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$

$$\log_a (x^k) = k \log_a x$$

The logs must have a consistent base.

i.e. You can move the power to the front.

Special cases:

$$\log_a a = 1 \quad (a > 0, a \neq 1)$$

$$\log_a 1 = 0 \quad (a > 0, a \neq 1)$$

$$\log\left(\frac{1}{x}\right) = \log(x^{-1}) = -\log(x)$$

We often try to avoid leaving fractions inside logs. So if the answer was:

$$\log_2\left(\frac{1}{3}\right)$$

You should write your answer as: $-\log_2 3$

Reciprocating the input negates the output.

Not in syllabus (but in MAT/PAT):

$$\log_a b = \frac{\log_c b}{\log_c a}$$

This is known as **changing the base**. So to get $\log_2 9$ in terms of log base 3:

$$log_2 9 = \frac{log_3 9}{log_3 2} = \frac{2}{log_3 2}$$

Examples

Write as a single logarithm:

a. $\log_3 6 + \log_3 7$

b. $\log_2 15 - \log_2 3$

c. $2\log_5 3 + 3\log_5 2$

d. $\log_{10} 3 - 4\log_{10}\left(\frac{1}{2}\right)$

Write in terms of $\log_a x$, $\log_a y$ and $\log_a z$

a. $\log_a(x^2yz^3)$

b. $\log_a\left(\frac{x}{y^3}\right)$

c. $\log_a\left(\frac{x\sqrt{y}}{z}\right)$

d. $\log_a\left(\frac{x}{a^4}\right)$

a ?

b ?

c ?

d ?

a ?

b ?

c ?

d ?

Anti Laws

These are **NOT LAWS OF LOGS**, but are mistakes students often make:

$$\log_a(b+c) = \log_a b + \log_a c$$

?

✘ FAIL

$$(\log_2 x)^3 = 3\log_2 x$$

?

✘ FAIL

Solving Equations with Logs

Solve the equation $\log_{10} 4 + 2\log_{10} x = 2$

This is a very common type of exam question.
The strategy is to **combine the logs into one** and isolate on one side.

?

Test Your Understanding

Edexcel C2 Jan 2013 Q6

Given that $2\log_2(x+15) - \log_2 x = 6$,

(*a*) show that $x^2 - 34x + 225 = 0$.

(5)

(*b*) Hence, or otherwise, solve the equation $2\log_2(x+15) - \log_2 x = 6$.

(2)

a

?

b

?

Exercise 14E

Pearson Pure Mathematics Year 1/AS
Pages 323-324

These are all strictly **non-calculator**!

Extension

1 *[AEA 2010 Q1b]* Solve the equation

$$\log_3(x-7) - \frac{1}{2}\log_3 x = 1 - \log_3 2$$

2 *[AEA 2008 Q5i]* Anna, who is confused about the rules of logarithms, states that

$$(\log_3 p)^2 = \log_3(p^2)$$
$$\log_3(p+q) = \log_3 p + \log_3 q$$

However, there is a value for p and a value for q for which both statements are correct. Find their values.

3 *[MAT 2007 1I]* Given that a and b are positive and

$$4(\log_{10} a)^2 + (\log_{10} b)^2 = 1$$

what is the greatest possible value of a?

4 *[MAT 2002 1F]* Observe that $2^3 = 8$, $2^5 = 32$, $3^2 = 9$ and $3^3 = 27$. From these facts, we can deduce that $\log_2 3$ is:

A) between $1\frac{1}{3}$ and $1\frac{1}{2}$

B) between $1\frac{1}{2}$ and $1\frac{2}{3}$

C) between $1\frac{2}{3}$ and 2

D) none of the above

Solutions on next slide.

Solutions to Extension Exercises

1 [AEA 2010 Q1b] Solve the equation

$$\log_3(x-7) - \frac{1}{2}\log_3 x = 1 - \log_3 2$$

?

2 [AEA 2008 Q5i] Anna, who is confused about the rules of logarithms, states that

$$(\log_3 p)^2 = \log_3(p^2)$$
$$\log_3(p+q) = \log_3 p + \log_3 q$$

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?

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$$4(\log_{10} a)^2 + (\log_{10} b)^2 = 1$$

what is the greatest possible value of a?

?

4 [MAT 2002 1F] Observe that $2^3 = 8$, $2^5 = 32$, $3^2 = 9$ and $3^3 = 27$. From these facts, we can deduce that $\log_2 3$ is:

A) between $1\frac{1}{3}$ and $1\frac{1}{2}$

B) between $1\frac{1}{2}$ and $1\frac{2}{3}$

C) between $1\frac{2}{3}$ and 2

D) none of the above

?

Solving equations with exponential terms

Solve $3^x = 20$

?

Solve $5^{4x-1} = 61$

?

Solving equations with exponential terms

Solve $3^x = 2^{x+1}$

Why can we not apply quite the same strategy here?

?

? Solution

Solving equations (disguised quadratic)

Solve the equation $5^{2x} - 12(5^x) + 20 = 0$, giving your answer to 3sf.

Test Your Understanding

1 Solve $3^{2x-1} = 5$, giving your answer to 3dp.

?

3 Solve $2^x 3^{x+1} = 5$, giving your answer in exact form.

?

2 Solve $3^{x+1} = 4^{x-1}$, giving your answer to 3dp.

?

Exercise 14F

Pearson Pure Mathematics Year 1/AS
Page 325

Extension

1 [MAT 2011 1H] How many *positive* values x which satisfy the equation:

$$x = 8^{\log_2 x} - 9^{\log_3 x} - 4^{\log_2 x} + \log_{0.5} 0.25$$

?

2 [MAT 2013 1J] For a real number x we denote by $[x]$ the largest integer less than or equal to x. Let n be a natural number. The integral

$$\int_0^n [2^x] \, dx$$

equals:

(A) $\log_2\left((2^n - 1)!\right)$
(B) $n\,2^n - \log_2\left((2^n)!\right)$
(C) $n\,2^n$
(D) $\log_2\left((2^n)!\right)$

(Warning: This one really is <u>very</u> challenging, even for MAT)

Solution to Extension Question 2

[MAT 2013 1J] For a real number x we denote by $[x]$ the largest integer less than or equal to x. Let n be a natural number. The integral

$$\int_0^n [2^x] \, dx$$

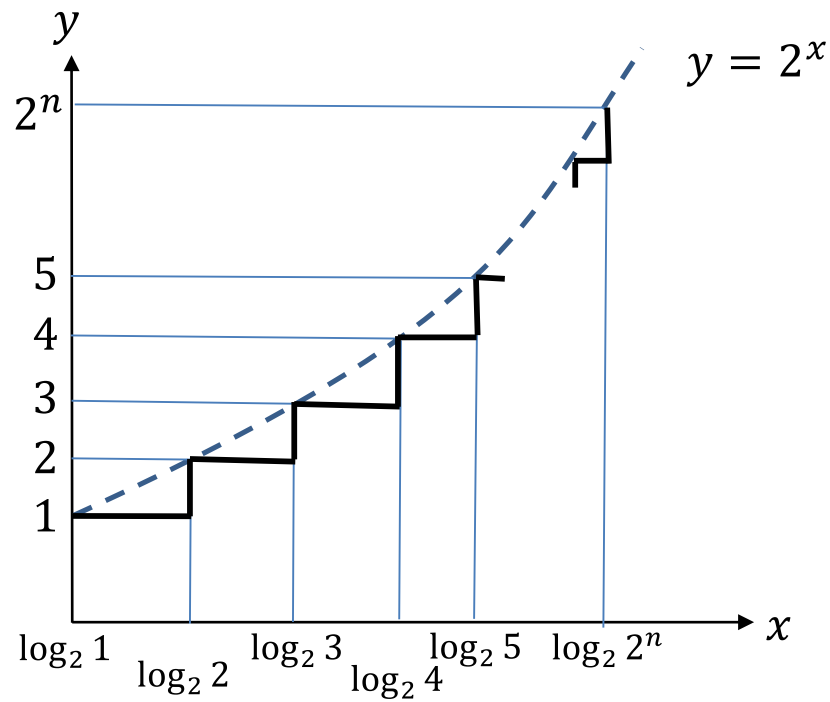
equals:

(A) $\log_2\left((2^n - 1)!\right)$

(B) $n\,2^n - \log_2\left((2^n)!\right)$

(C) $n\,2^n$

(D) $\log_2\left((2^n)!\right)$



This biggest challenge is sketching the graph! Because of the rounding down, the graph jumps up 1 at a time, giving a bunch of rectangles. We can use logs to find the corresponding x values at which these jumps occur, which progressively become closer and closer together. The last y value is 2^n, thus the last x value is $\log_2 2^n = n$.

The area, using the rectangles, is thus:

$$\begin{aligned}
&1(\log_2 2 - \log_2 1) + 2(\log_2 3 - \log_2 2) + 3(\log_2 4 - \log_2 3) + \cdots + (2^n - 1)(\log_2 2^n - \log_2(2^n - 1)) \\
&= \log_2 2 - \log_2 1 + 2\log_2 3 - 2\log_2 2 + 3\log_2 4 - 3\log_2 3 + \cdots + (2^n - 1)\log_2 2^n - (2^n - 1)\,log_2(2^n - 1) \\
&= -\log_2 2 - \log_2 3 - \cdots - \log_2(2^n - 1) + (2^n - 1)\,log_2 2^n \\
&= -(\log_2\left(2 \times 3 \times 4 \times \cdots \times (2^n - 1)\right) + n(2^n - 1) \\
&= n\,2^n - log_2\left((2^n)!\right)
\end{aligned}$$

Natural Logarithms

We have previously seen that $y = \log_a x$ is the inverse function of $y = a^x$.
We also saw that e^x is "**the**" exponential function.
The inverse of e^x is $\log_e x$, but because of its special importance, it has its own function name!

✎ The inverse of $y = e^x$ is $y = \ln x$

$$\ln e^x = ?$$
$$e^{\ln x} = ?$$

Since "e to the power of" and "ln of" are inverse functions, they cancel each other out!

Solve $e^x = 5$

?

Solve $2 \ln x + 1 = 5$

?

Quadratics in e^x

In previous chapters we've already dealt with quadratics in disguise, e.g. "quadratic in sin". We therefore just apply our usual approach: either make a suitable substitution so the equation **is** then quadratic, or (strongly recommended!) **go straight for the factorisation**.

Solve $e^{2x} + 2e^x - 15 = 0$

?

Solve $e^x - 2e^{-x} = 1$

?

Test Your Understanding

Solve $\ln(3x+1) = 2$

?

Solve $e^{2x} + 5e^{x} = 6$

?

Solve $2^{x}e^{x+1} = 3$ giving your answer as an exact value.

?

Exercise 14G

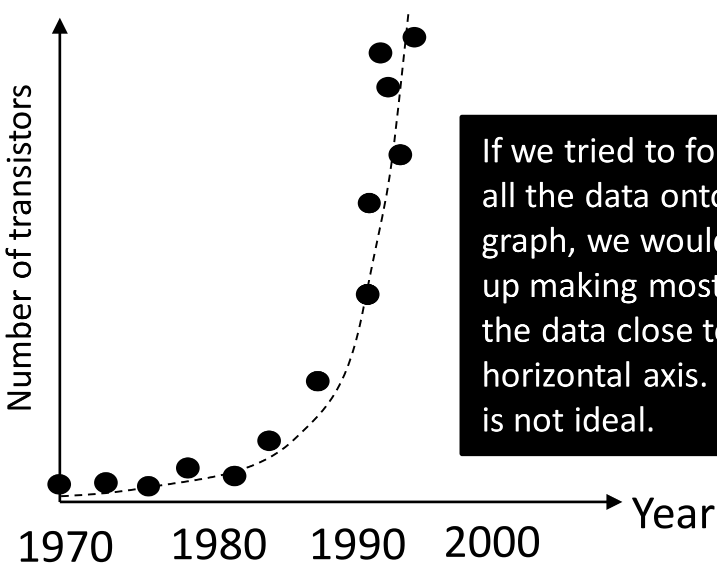
Pearson Pure Mathematics Year 1/AS
Page 327-328

Graphs for Exponential Data

In Science and Economics, **experimental data often has exponential growth**, e.g. bacteria in a sample, rabbit populations, energy produced by earthquakes, my Twitter followers over time, etc.

Because exponential functions increase rapidly, it tends to look a bit rubbish if we tried to draw a suitable graph:

Take for example "Moore's Law", which hypothesised that the processing power of computers would double every 2 years. Suppose we tried to plot this for computers we sampled over time:

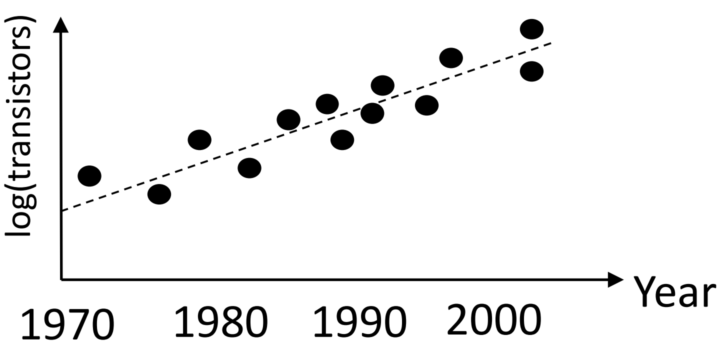


If we tried to force all the data onto the graph, we would end up making most of the data close to the horizontal axis. This is not ideal.

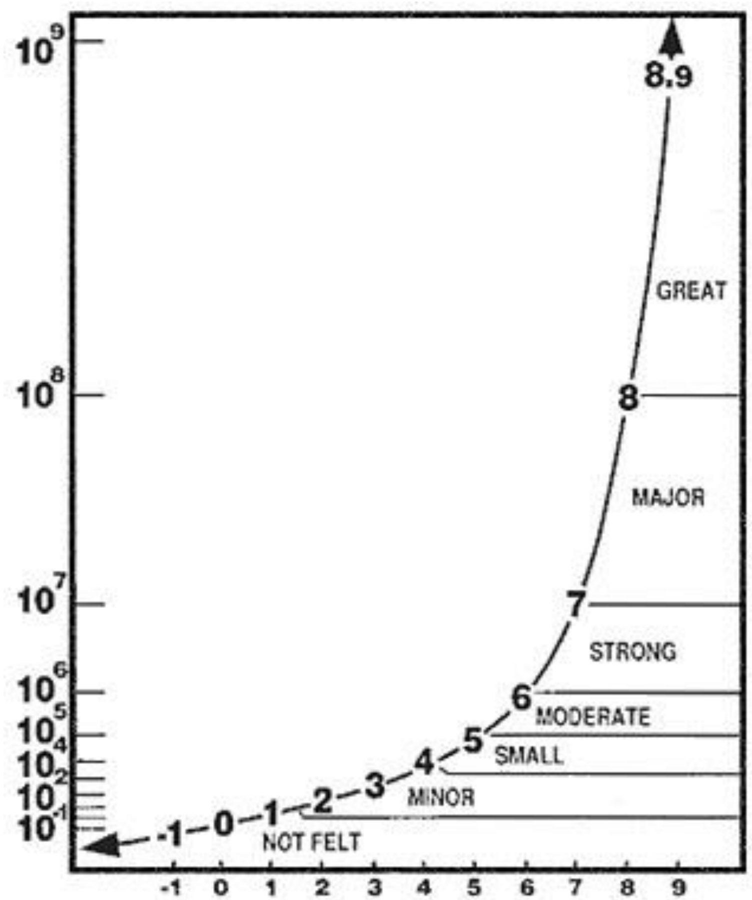
But suppose we **took the log** of the number of transistors for each computer. Suppose the number of transistors one year was y, then doubled 2 years later to get $2y$.
When we log (base 2) these:

$$y \rightarrow \log_2 y$$
$$2y \rightarrow \log_2(2y) = \log_2 2 + \log_2 y$$
$$= 1 + \log_2 y$$

The logged value only increased by 1! Thus **taking the log of the values turns <u>exponential growth</u> into <u>linear growth</u> (because each time we would have doubled, we're now just adding 1), and the resulting graph is a straight line.**



Graphs for Exponential Data



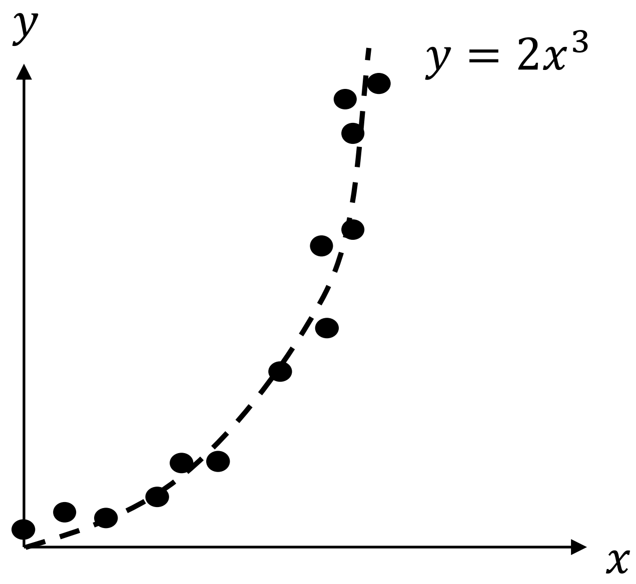
Richter Scale

Because the energy involved in **earthquakes** decreases exponentially from the epicentre of the earthquake, such energy values recorded from different earthquakes would **vary wildly**.

The **Richter Scale** is a **logarithmic scale**, and takes the log (base 10) of the amplitude of the waves, giving a more even spread of values in a more sensible range.
(The largest recorded value on the Richter Scale is 9.5 in Chile in 1960, and 15 would destroy the Earth completely – evil scientists take note)

The result is that an earthquake just 1 greater on the Richter scale would in fact be 10 times as powerful.

Other Non-Linear Growth



We would also have similar graphing problems if we tried to plot data that followed some **polynomial function** such as a quadratic or cubic.

We will therefore look at the process to convert a **polynomial graph into a linear one**, as well as a **exponential graph into a linear one…**

Turning non-linear graphs into linear ones

Case 1: Polynomial → Linear

Suppose our original model was a polynomial one*:

$$y = ax^n$$

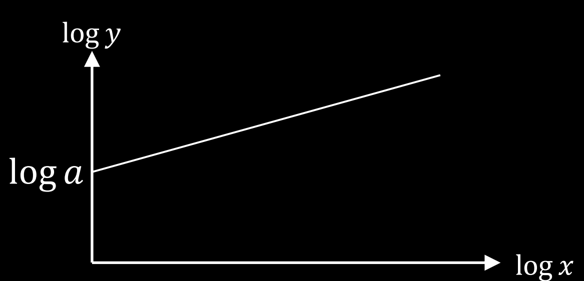
Then taking logs of both sides:

$$\log y = \log ax^n$$
$$\log y = \log a + n \log x$$

We can compare this against a straight line:

$$Y = mX + c$$

✎ If $y = ax^n$, then the graph of $\log y$ against $\log x$ will be a straight line wih gradient n and vertical intercept $\log a$.



* We could also allow non-integer n; the term would then not strictly be polynomial, but we'd still say the function had "polynomial growth".

Case 2: Exponential → Linear

Suppose our original model was an exponential one:

$$y = ab^x$$

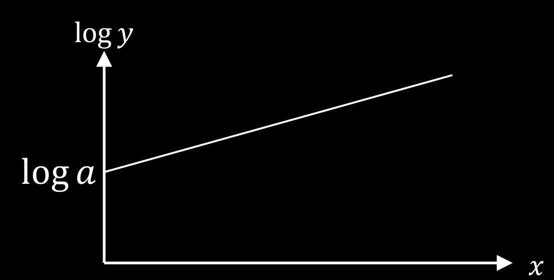
Then taking logs of both sides:

$$\log y = \log ab^x$$
$$\log y = \log a + x \log b$$

Again we can compare this against a straight line:

$$Y = mX + c$$

✎ If $y = ab^x$, then the graph of $\log y$ against x will be a straight line with gradient $\log b$ and vertical intercept $\log a$.



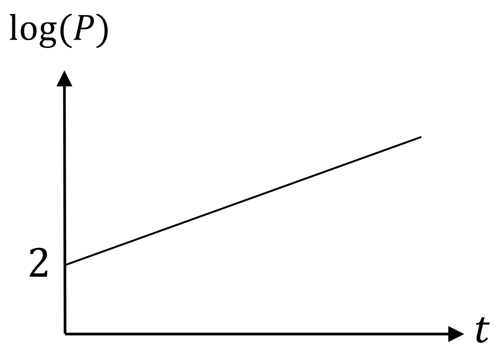
The key difference compared to Case 1 is that we're **only logging the y values** (e.g. number of transistors), not the x values (e.g. years elapsed). **Note that you <u>do not need to memorise</u> the contents of these boxes and we will work out from scratch each time…**

In summary, logging the y-axis **turns an exponential graph into a linear one**. Logging **both** the x and y-axis turns a polynomial graph into a linear one.

Example

[Textbook] The graph represents the growth of a population of bacteria, P, over t hours. The graph has a gradient of 0.6 and meets the vertical axis at $(0,2)$ as shown.
A scientist suggest that this growth can be modelled by the equation $P = ab^t$, where a and b are constants to be found.

a. Write down an equation for the line.
b. Using your answer to part (a) or otherwise, find the values of a and b, giving them to 3 sf where necessary.
c. Interpret the meaning of the constant a in this model.



a ?

b ?

c ?

Example

[Textbook] The table below gives the rank (by size) and population of the UK's largest cities and districts (London is number 1 but has been excluded as an outlier).

City	B'ham	Leeds	Glasgow	Sheffield	Bradford
Rank, R	2	3	4	5	6
Population, P	1 000 000	730 000	620 000	530 000	480 000

The relationship between the rank and population can be modelled by the formula:
$P = aR^n$ where a and n are constants.

a) Draw a table giving values of $\log R$ and $\log P$ to 2dp.

b) Plot a graph of $\log R$ against $\log P$ using the values from your table and draw the line of best fit.

c) Use your graph to estimate the values of a and n to two significant figures.

Textbook Error: They use $R = aP^n$ but then plot $\log P$ against $\log R$.

a

?

b

?

c

?

Test Your Understanding

Dr Frost's wants to predict his number of Twitter followers P (@DrFrostMaths) t years from the start 2015. He predicts that his followers will increase exponentially according to the model $P = ab^t$, where a, b are constants that he wishes to find.

He records his followers at certain times. Here is the data:

Years t after 2015:	0.7	1.3	2.2
Followers P:	2353	3673	7162

a) Draw a table giving values of t and $\log P$ (to 3dp).

b) A line of best fit is drawn for the data in your new table, and it happens to go through the first data point above (where $t = 0.7$) and last (where $t = 2.2$). Determine the equation of this line of best fit. (The y-intercept is 3.147)

c) Hence, determine the values of a and b in the model.

d) Estimate how many followers Dr Frost will have at the start of 2020 (when $t = 5$).

a

t	?
$\log P$	

b ?

c ?

d ?

Reflections: Consider what we're doing in this whole process in case you don't understand why we're doing all of this:

1. We want to find the **parameters of a model**, e.g. $P = ab^t$ that **best fits the data** (in this case the parameters we want to find are a and b).
2. If the data had a linear trend, then this would be easy! We know from KS3 that we'd just plot the data, find the line of best fit, then use the gradient and y-intercept to work out the m and c in our linear model.
3. But the original data wasn't linear, and it would be harder to draw an 'exponential curve of best fit'.
4. We therefore log the model so that the plotted data then roughly forms a straight line, and then we can then draw a (straight) line of best fit.
5. The gradient and y-intercept of this line then allows us to estimate the parameters a and b in the original model that best fit the data.

The process of finding parameters in a model, that best fits the data, is known as **regression**.

Exercise 14H

Pearson Pure Mathematics Year 1/AS
Page 331-333

The End