P2 Chapter 8 :: Parametric Equations

jfrost@tiffin.kingston.sch.uk

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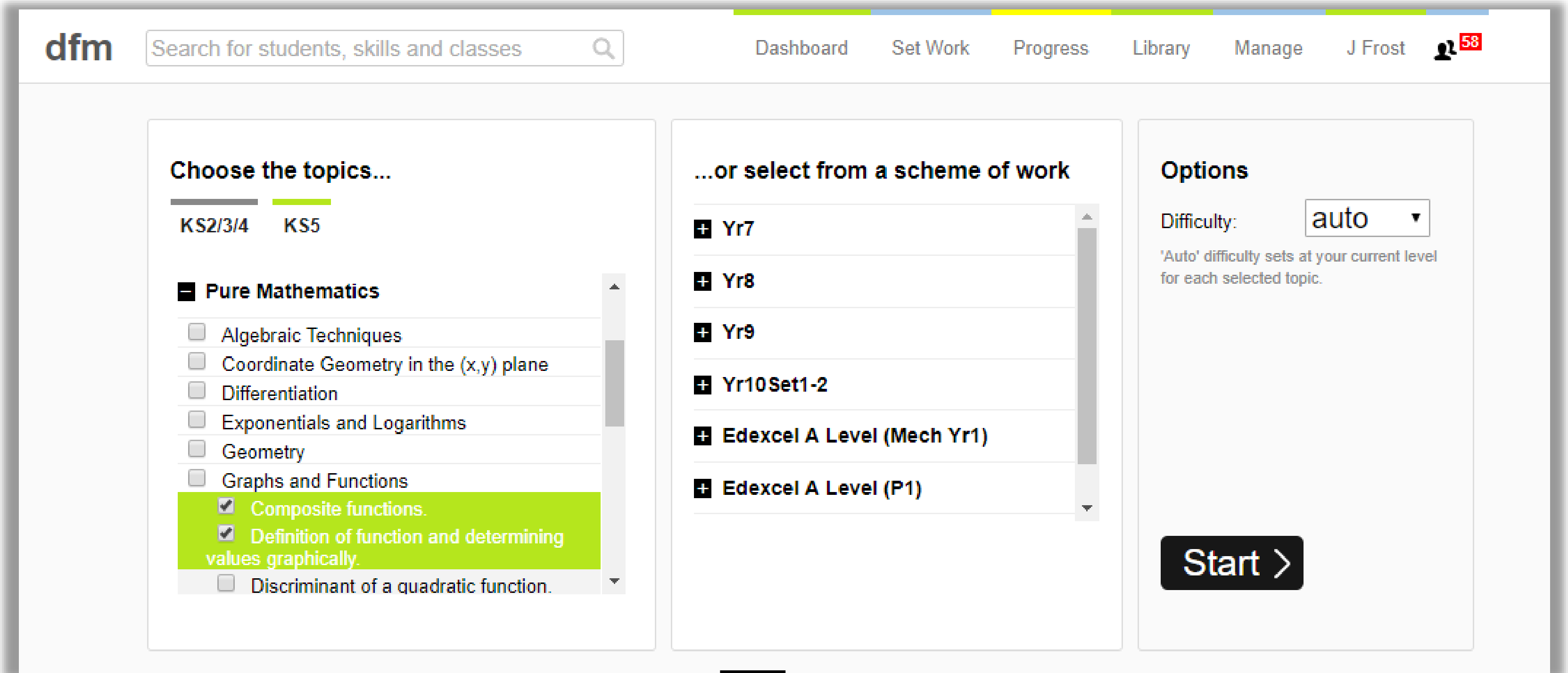

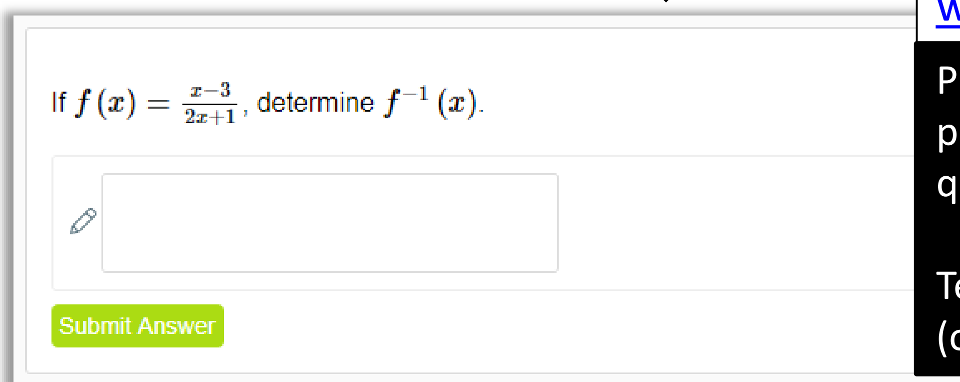

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Chapter Overview

This chapter is very similar to the trigonometry chapters in Year 1. The only difference is that new trig functions: sec, $cosec$ and cot, are introduced.

1:: Converting from parametric to Cartesian form.

If $x = 2\cos t + 1$ and $y = 3\sin t$, find a Cartesian equations connecting x and y.

2:: Sketching parametric curves.

Sketch the curve with parametric equations $x = 2t$ and $y = \frac{5}{t}$.

3:: Finding points of intersection.

Curve C_1 has the parametric equations $x = t^2$ and $y = 4t$. The curve C_2 has the Cartesian equation $x + y + 4 = 0$. The two curves intersect at A. Find the coordinates of A.

4:: Modelling

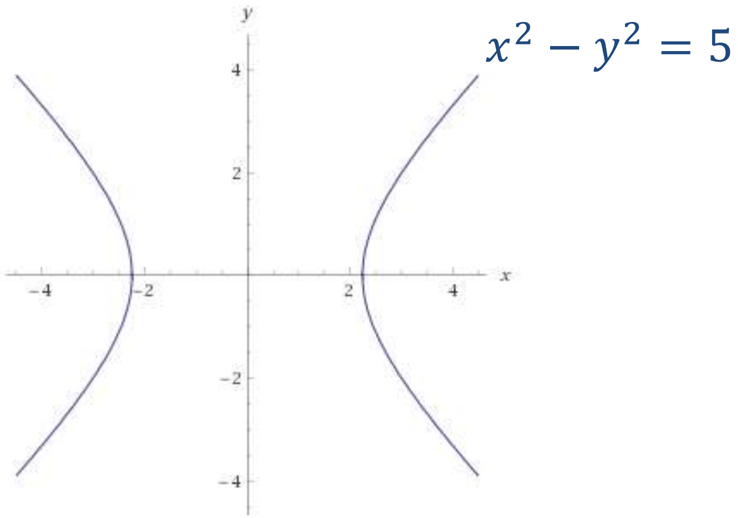
A plane's position at time t seconds after take-off can be modelled with the parametric equations:

$x = (v\cos\theta)t$ m, $\quad y = (v\sin\theta)t$ m, $\quad t > 0$

…

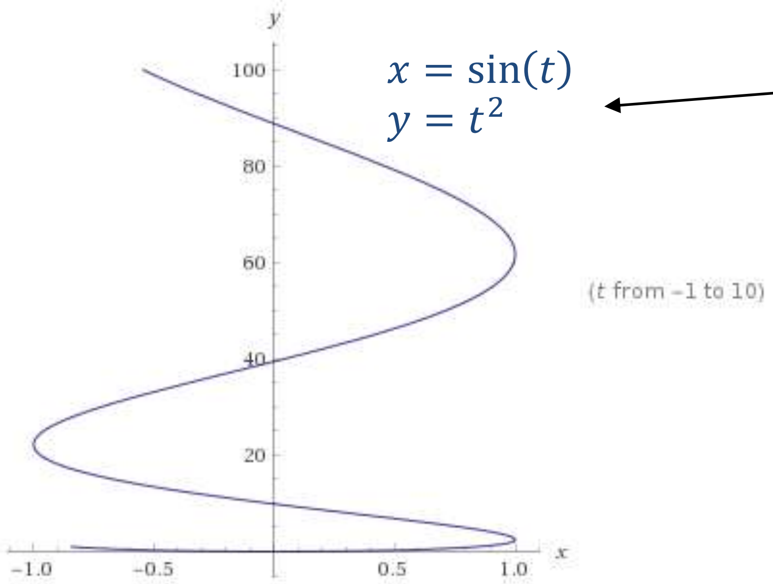
Teacher Note: There is no change in this chapter relative to the old pre-2017 syllabus.

What are they and what is the point?



Typically, with two variables x and y, we can relate the two by **a single equation involving just x and y**.
This is known as a **<u>Cartesian equation</u>**.

The line shows all points (x, y) which satisfy the Cartesian equation.



However, in Mechanics for example, we might want each of the x and y values to be some function of time t, as per this example.

This would allow us to express the position of a particle at time t as the vector:

$$\begin{pmatrix} \sin t \\ t^2 \end{pmatrix}$$

These are known as **parametric equations**, because **each of x and y are defined in terms of some other variable, known as the parameter** (in this case t).

Converting parametric to Cartesian

How could we convert these parametric equations into a single Cartesian one?

$$x = 2t, \qquad y = t^2, \qquad -3 < t < 3$$

?

What is the domain of the function?

If $x = p(t)$ and $y = q(t)$ can be written as $y = f(x)$, then the domain of f is the range of p...

?

and the range of f is the range of q.

?

Further Example

[Textbook] A curve has the parameter equations

$$x = \ln(t+3), \qquad y = \frac{1}{t+5}, \qquad t > -2$$

a) Find a Cartesian equation of the curve of the form $y = f(x)$, $x > k$, where k is a constant to be found.
b) Write down the range of $f(x)$.

? a

? b

A common strategy for domain/range questions is to consider what happens are the boundary value (in this case -2), then since $t > -2$, consider what happens as t increases.

Test Your Understanding

Edexcel C4 Jan 2008 Q7

The curve C has parametric equations

$$x = \ln(t+2), \quad y = \frac{1}{(t+1)}, \quad t > -1.$$

(c) Find a cartesian equation of the curve C, in the form $y = \text{f}(x)$. **(4)**

?

Edexcel C4 Jan 2011

6. The curve C has parametric equations

$$x = \ln t, \quad y = t^2 - 2, \quad t > 0.$$

(b) a cartesian equation of C.

(3)

?

Exercise 8A

Pearson Pure Mathematics Year 2/AS
Pages 200-202

…when you have trig identities

When we have trig functions we have to use identities to find the Cartesian equation. Generally we use $\sin^2 t + \cos^2 t \equiv 1$ or $1 + \tan^2 t \equiv \sec^2 t$

[Textbook] A curve has the parametric sequences $x = \sin t + 2$, $y = \cos t - 3$, $t \in \mathbb{R}$.

a) Find a Cartesian equation for the curve.

b) Hence sketch the curve.

?

[Textbook] A curve is defined by the parametric equations

$$x = \sin t\,, \qquad y = \sin 2t\,, \qquad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

a) Find a Cartesian equation of the curve in the form $y = f(x)$, $-k \leq x \leq k$, stating the value of the constant k.

b) Write down the range of $f(x)$.

a

?

b

?

Test Your Understanding

C4 June 2013

which double angle formula would be best here?

4. A curve C has parametric equations

$$x = 2\sin t, \qquad y = 1 - \cos 2t, \qquad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

(*b*) Find a cartesian equation for C in the form

$$y = f(x), \qquad -k \leq x \leq k,$$

stating the value of the constant k.

?

[Textbook] A curve C has parametric equations

$$x = \cot t + 2, \qquad y = cosec^2\, t - 2, \qquad 0 < t < \pi$$

a) Find the equation of the curve in the form $y = f(x)$ and state the domain of x for which the curve is defined.

b) Hence, sketch the curve.

? a

? b

Exercise 8B

Pearson Pure Mathematics Year 2/AS
Pages 204-206

Further Exam Practice

C4 June 2012 Q6

Figure 2 shows a sketch of the curve C with parametric equations

$$x = \sqrt{3}\sin 2t, \qquad y = 4\cos^2 t, \qquad 0 \leq t \leq \pi.$$

(*c*) Find a cartesian equation of C.

(3)

?

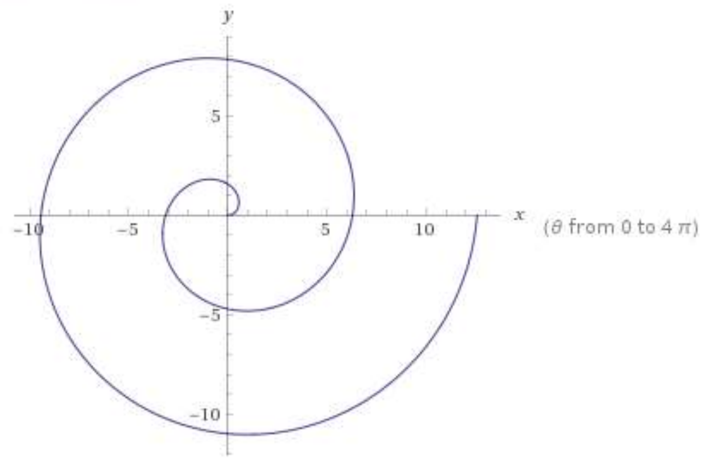
Sketching Parametric Curves

We saw that one strategy for sketching parametric curves is to convert into a Cartesian equation, and hope this is a form we recognise (e.g. quadratic or equation of circle) to appropriately sketch.
However, some parametric equations can't easily be turned into Cartesian form:

Input interpretation:

plot	$x = \theta \cos(\theta)$ $y = \theta \sin(\theta)$	$\theta = 0$ to 4π

Parametric plot:



These parametric equations in Cartesian form would be $\sqrt{x^2 + y^2} = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)$; this would obviously be incredibly hard to sketch!

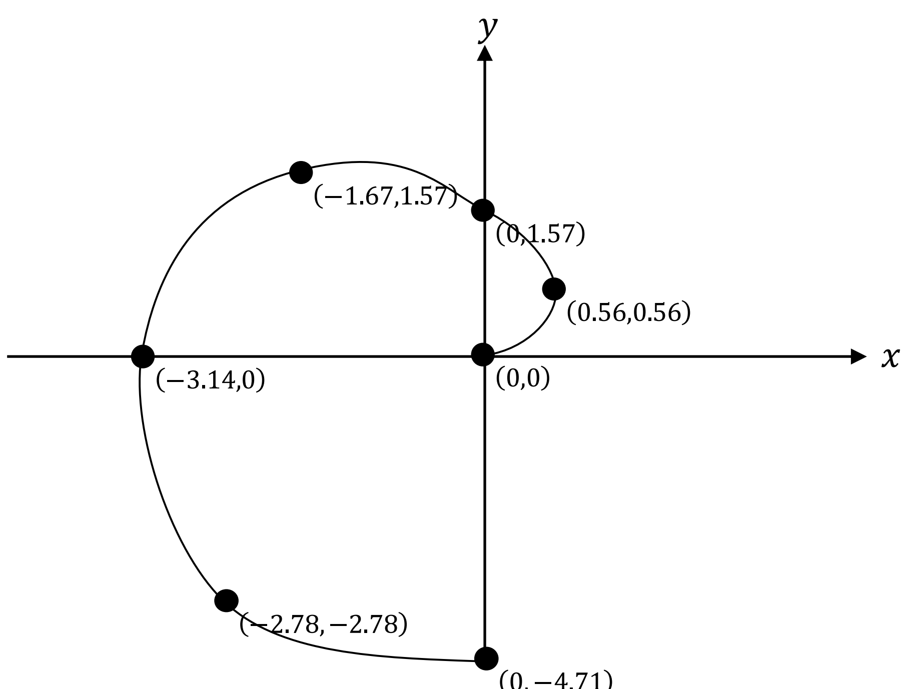
Instead we can try different values of t and determine the point (x, y) for each value to get a sequence of points…

Sketching Parametric Curves

Input interpretation:

plot $x = \theta\cos(\theta)$, $y = \theta\sin(\theta)$, $\theta = 0$ to 4π

$\boldsymbol{\theta}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
$\boldsymbol{x}$ / $\boldsymbol{y}$	?	?	?	?	?	?	?	?	?



Test Your Understanding

[Textbook] Draw the curve given by the parametric equations $x = 2t$, $y = t^2$, for $-1 \leq t \leq 5$.

?

Exercise 8C

Pearson Pure Mathematics Year 2/AS
Pages 207-208

(This exercise could probably be skipped for classes in a rush)

Points of Intersection

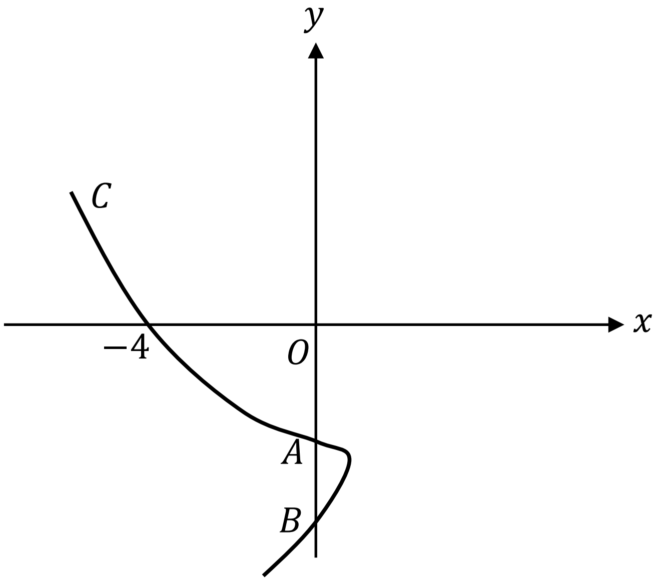
We can find where a parametric curve crosses a particular axis or where curves cross each other.

The key is to first find the value of the parameter t.

[Textbook] The diagram shows a curve C with parametric equations $x = at^2 + t, \quad y = a(t^3 + 8)$, $t \in \mathbb{R}$, where a is a non-zero constant. Given that C passes through the point $(-4,0)$,

a) find the value of a.

b) find the coordinates of the points A and B where the curve crosses the y-axis.



a

?

b

?

Points of Intersection

[Textbook] A curve is given parametrically by the equations $x = t^2,\ y = 4t$. The line $x + y + 4 = 0$ meets the curve at A. Find the coordinates of A.

$$x + y + 4 = 0$$
$$\therefore t^2 + 4t + 4 = 0$$
$$(t+2)^2 = 0 \quad \therefore \quad t = -2$$
$$\therefore x = (-2)^2 = 4$$
$$y = 4(-2) = -8$$
$$\mathrm{A}(4, -8)$$

Whenever you want to solve a Cartesian equation and pair of parametric equations simultaneously, substitute the parametric equations into the Cartesian one.

[Textbook] The diagram shows a curve C with parametric equations

$$x = \cos t + \sin t, \qquad y = \left(t - \frac{\pi}{6}\right)^2, \qquad -\frac{\pi}{2} < t < \frac{4\pi}{3}$$

a) Find the point where the curve intersects the line $y = \pi^2$.
b) Find the coordinates of the points A and B where the curve cuts the y-axis.

a

?

?

x

Test Your Understanding

C4 Jan 2013

5.

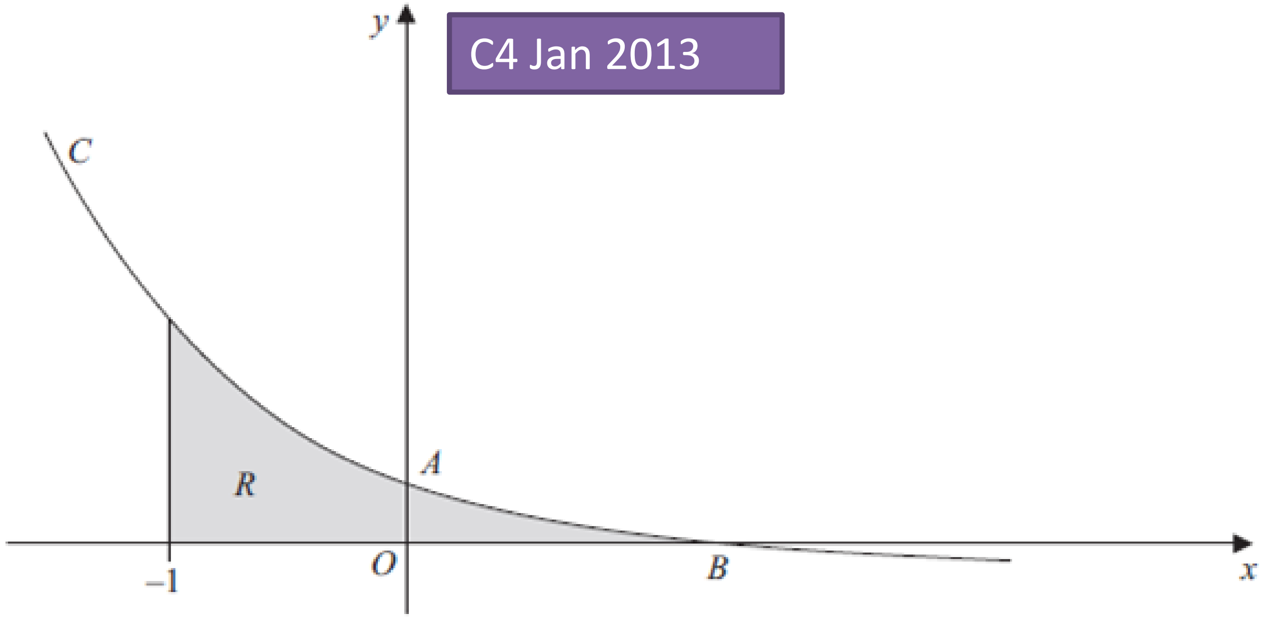


Figure 2

Figure 2 shows a sketch of part of the curve C with parametric equations

$$x = 1 - \frac{1}{2}t, \qquad y = 2^t - 1.$$

The curve crosses the y-axis at the point A and crosses the x-axis at the point B.

(*a*) Show that A has coordinates (0, 3).

(2)

(*b*) Find the x-coordinate of the point B.

(2)

a ?

b ?

Exercise 8D

Pearson Pure Mathematics Year 2/AS

Pages 211-213

Modelling

As we saw at the start of this chapter, parametric equations are frequently used in mechanics, particularly where the (x, y) position (the Cartesian variables) depends on time t (the parameter).

[Textbook] A plane's position at time t seconds after take-off can be modelled with the following parametric equations:

$$x = (v\cos\theta)t\ \text{m}, \qquad y = (v\sin\theta)t\ \text{m}, \qquad t > 0$$

where v is the speed of the plane, θ is the angle of elevation of its path, x is the horizontal distance travelled and y is the vertical distance travelled, relative to a fixed origin.

When the plane has travelled 600m horizontally, it has climbed 120m.

a. find the angle of elevation, θ.

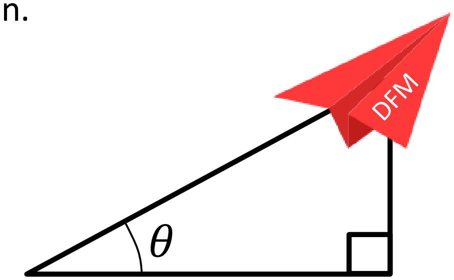
Given that the plane's speed is 50 m s^{-1},

b. find the parametric equations for the plane's motion.

c. find the vertical height of the plane after 10 seconds.

d. show that the plane's motion is a straight line.

e. explain why the domain of t, $t > 0$, is not realistic.



a ?

b ?

c ?

d ?

e ?

Further Example

[Textbook] The motion of a figure skater relative to a fixed origin, O, at time t minutes is modelled using the parametric equations

$$x = 8\cos 20t\,, \qquad y = 12\sin\left(10t - \frac{\pi}{3}\right), \qquad t \geq 0$$

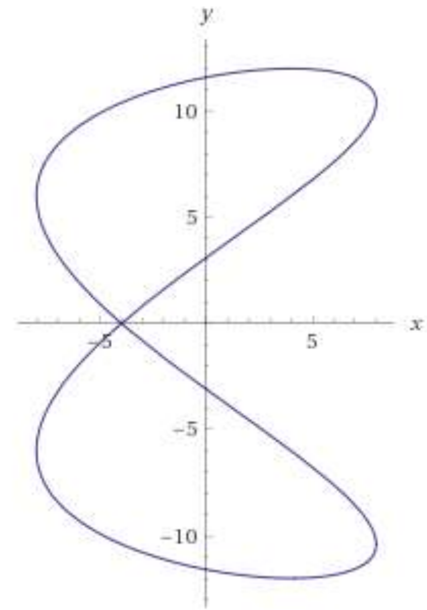
where x and y are measured in metres.

a) Find the coordinates of the figure skater at the beginning of his motion.

b) Find the coordinates of the point where the figure skater intersects his own path.

c) Find the coordinates of the points where the path of the figure skater crosses the y-axis.

d) Determine how long it takes the figure skater to complete one complete figure-of-eight motion.



a ?

b ?

c ?

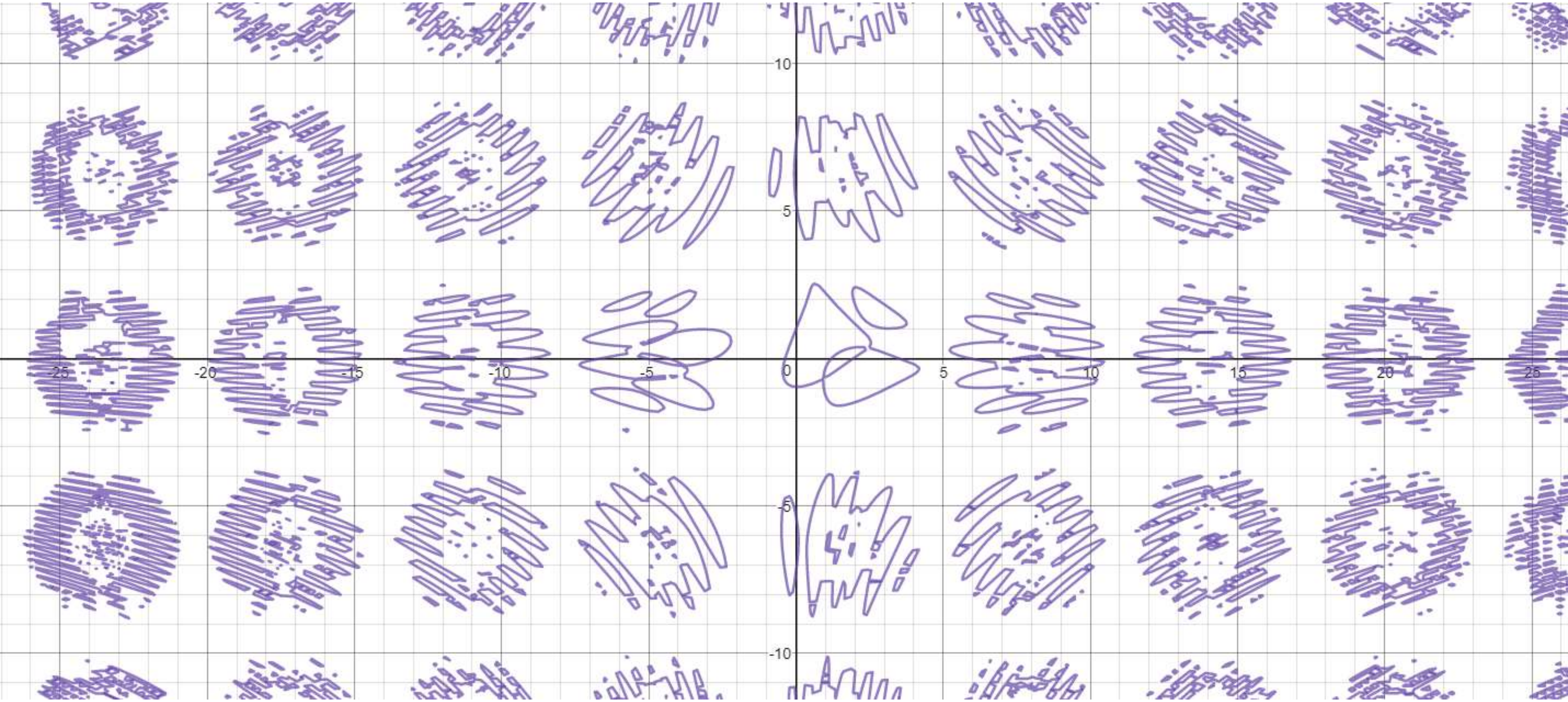
d ?

Exercise 8E

Pearson Pure Mathematics Year 2/AS
Pages 218-220

Just For Fun…

$$\sin(\sin(x) + \cos(y)) = \cos(\sin(xy) + \cos(x))$$



(Yes, I realise this is a Cartesian equation, not parametric ones – but it looked pretty!)