Chapter 8 - Mechanics

Further Kinematics

Chapter Overview

1. Vectors in Kinematics
2. Vector Methods with Projectiles
3. Variable Acceleration in One Dimension
4. Differentiating Vectors
5. Integrating Vectors

Topics	What students need to learn:		
	Content		Guidance
7 **Kinematics**	7.1	**Understand and use the language of kinematics: position; displacement; distance travelled; velocity; speed; acceleration.**	Students should know that distance and speed must be positive.
	7.2	**Understand, use and interpret graphs in kinematics for motion in a straight line: displacement against time and interpretation of gradient; velocity against time and interpretation of gradient and area under the graph.**	**Graphical solutions to problems may be required.**
	7.3	**Understand, use and derive the formulae for constant acceleration for motion in a straight line.** Extend to 2 dimensions using vectors.	**Derivation may use knowledge of sections 7.2 and/or 7.4** **Understand and use *suvat* formulae for constant acceleration in 2-D,** e.g. $\mathbf{v} = \mathbf{u} + \mathbf{a}t$, $\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ with vectors given in $\mathbf{i} - \mathbf{j}$ or column vector form. Use vectors to solve problems.

7.4	**Use calculus in kinematics for motion in a straight line:** $v = \frac{dr}{dt}$, $a = \frac{dv}{dt} = \frac{d^2r}{dt^2}$ $\mathbf{r} = \int \mathbf{v} \ dt$, $\mathbf{v} = \int \mathbf{a} \ dt$ Extend to 2 dimensions using vectors.	**The level of calculus required will be consistent with that in Sections 7 and 8 in Paper 1** and Sections 6 and 7 in Paper 2. Differentiation and integration of a vector with respect to time. e.g. Given $\mathbf{r} = t^2\mathbf{i} + t^{\frac{3}{2}}\mathbf{j}$, find $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ at a given time.

1. Vectors in Kinematics

If a particle starts from the point with position vector $\boldsymbol{r}_0$, and moves with constant velocity $\boldsymbol{v}$, its displacement from its initial position at time t is given by $\boldsymbol{v}$t and it position vector $\boldsymbol{r}$ is given by:

Example

At time $t = 0$, where t is the time (in seconds), a particle is at the point with position vector $(4\boldsymbol{i} - \boldsymbol{j})$ m and travels with velocity $(-2\boldsymbol{i} + 2\boldsymbol{j})$ ms^{-1}. Find:

a) The position vector of the particle after t seconds

b) The distance the particle is from the origin, O, after 3 seconds.

Example

A particle starts at a point 8m from O at an angle of 45^{O} anti-clockwise from east and travels with a velocity $(-2\boldsymbol{i} - 3\boldsymbol{j})$ ms^{-1}, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors due east and north respectively.

Find the position vector of the particle after t seconds in the form $\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{v}$.

Example – Using SUVAT with Vectors

A particle is initially travelling with velocity $(-2\boldsymbol{i} - 9\boldsymbol{j})$ ms^{-1} and 2 seconds later it has a velocity of $(6\boldsymbol{i} - 11\boldsymbol{j})$ ms^{-1}, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the directions of the positive x- and y- axes respectively. Given that the acceleration of the particle is constant, find:

a) The acceleration

b) The magnitude of the acceleration

c) The angle that the acceleration makes with the vector $\boldsymbol{j}$

Example *(Textbook p161 Example 3)*

An ice skater is skating on a large flat ice rink. At time $t = 0$ the skater is at a fixed point O and is travelling with velocity $(2.4\boldsymbol{i} - 0.6\boldsymbol{j})$ ms^{-1}.
At time $t = 20$ s the skater is travelling with velocity $(-5.6\boldsymbol{i} + 3.4\boldsymbol{j})$ ms^{-1}.
Relative to O, the skater has position vector $\boldsymbol{s}$ at time t seconds.
Modelling the ice skater as a particle with constant acceleration, find:

(a) The acceleration of the ice skater
(b) An expression for $\boldsymbol{s}$ in terms of t
(c) The time at which the skater is directly north-east of O.

A second skater travels so that she has position vector $\boldsymbol{r} = (1.1t - 6)\boldsymbol{j}$ m relative to O at time t.

(d) Show that the two skaters will meet.

Test Your Understanding *(EdExcel M1 May 2013(R) Q6)*

[In this question **i** *and* **j** *are horizontal unit vectors due east and due north respectively. Position vectors are given with respect to a fixed origin O.]*

A ship S is moving with constant velocity $(3\mathbf{i} + 3\mathbf{j})$ km h^{-1}. At time $t = 0$, the position vector of S is $(-4\mathbf{i} + 2\mathbf{j})$ km.

(a) Find the position vector of S at time t hours.

(2)

A ship T is moving with constant velocity $(-2\mathbf{i} + n\mathbf{j})$ km h^{-1}. At time $t = 0$, the position vector of T is $(6\mathbf{i} + \mathbf{j})$ km. The two ships meet at the point P.

(b) Find the value of n.

(5)

(c) Find the distance OP.

(4)

Exercise 7A Page 162

2. Vector Methods with Projectiles

Previously we considered the initial speed of the projectile and the angle of projection. But we could also **use a velocity vector to represent the initial projection** (vectors have both direction and magnitude) and subsequent motion.

Example

A ball is projected from the origin with velocity $(12\boldsymbol{i} + 24\boldsymbol{j})$ms^{-1} where $\boldsymbol{i}$ and $\boldsymbol{j}$ are horizontal and vertical unit vectors respectively. The particle moves freely under gravity. Find:

a) The position vector of the ball after 3s

b) The speed of the ball after 3s

c) The ball strikes the ground at point B. Determine the distance OB

Example

A particle *P* is projected with velocity $(4p\boldsymbol{i} + 5p\boldsymbol{j})$ ms^{-1} from a point *O* on a horizontal plane, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are horizontal and vertical unit vectors respectively.
The particle *P* strikes the plane at the point *A,* which is 800 m from *O*.

a) Show that $p = 14$.

b) Find the time of flight from *O* to *A*.

The particle *P* passes through a point *B* with speed 60 m s^{-1}.

c) Find the height of *B* above the horizontal plane.

Test Your Understanding *(EdExcel M2 Jan 2012 Q7)*

[*In this question, the unit vectors* **i** *and* **j** *are horizontal and vertical respectively.*]

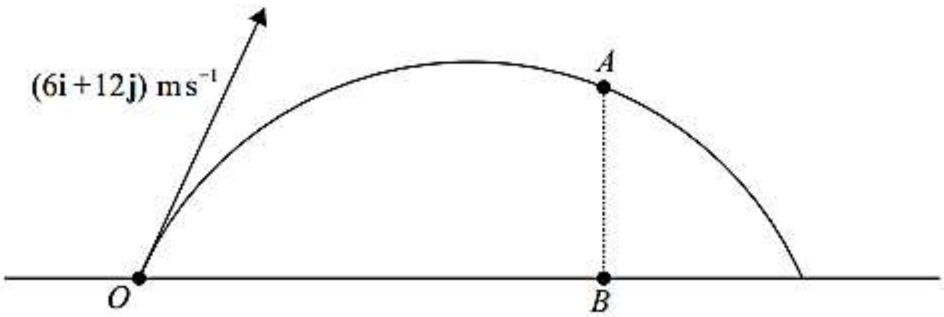


Figure 3

The point O is a fixed point on a horizontal plane. A ball is projected from O with velocity $(6\mathbf{i}+12\mathbf{j})$ m s^{-1}, and passes through the point A at time t seconds after projection. The point B is on the horizontal plane vertically below A, as shown in Figure 3. It is given that $OB = 2AB$.

Find

(a) the value of t,

(7)

(b) the speed, V m s^{-1}, of the ball at the instant when it passes through A.

(5)

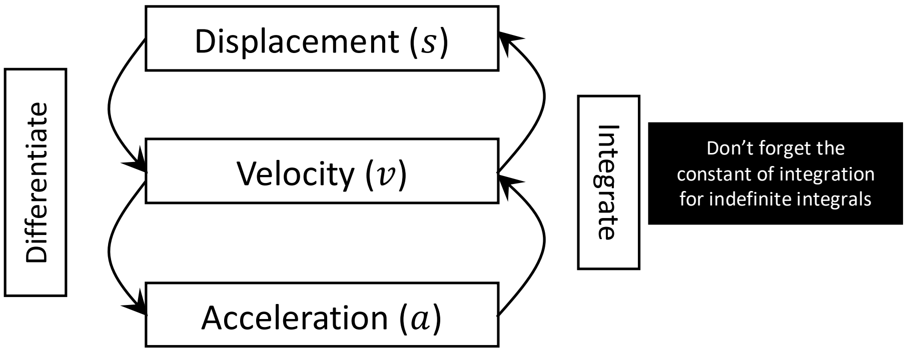
At another point C on the path the speed of the ball is also V m s^{-1}.

(c) Find the time taken for the ball to travel from O to C.

(3)

Exercise 8B Page 166

3. Variable Acceleration in One Dimension



Example

A particle is moving in a straight line with acceleration at time t seconds given by

$$a = \cos 2\pi t \text{ ms}^{-2}, \quad t \geq 0$$

The velocity of the particle at time $t = 0$ is $\frac{1}{2\pi}$ ms^{-1}. Find:

a) an expression for the velocity at time t seconds
b) the maximum speed
c) the distance travelled in the first 3 seconds.

Test Your Understanding *(Textbook p168 Example 6)*

A particle of mass 6kg is moving on the positive x-axis. At time t seconds the displacement, s, of the particle from the origin is given by

$$s = 2t^{\frac{3}{2}} + \frac{e^{-2t}}{3} \text{ m}, \quad t \geq 0$$

a) Find the velocity of the particle when $t = 1.5$.
b) Given that the particle is acted on by a single force of variable magnitude F N which acts in the direction of the positive x-axis,
c) Find the value of F when $t = 2$

Exercise 8C Page 168

4. Differentiating Vectors

We use calculus with 2-d (and 3-d) vectors by differentiating and integrating each function of time separately:

If $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$, then

Example

A particle P of mass 0.8kg is acted on by a single force $\mathbf{F}$ N. Relative to a fixed origin O, the position vector of P at time t seconds is $\boldsymbol{r}$ metres, where

$$\boldsymbol{r} = 2t^3\boldsymbol{i} + 50t^{-\frac{1}{2}}\boldsymbol{j}, \quad t \geq 0$$

Find:

a) the speed of P when $t = 4$
b) the acceleration of P as a vector when $t = 2$
c) $\mathbf{F}$ when $t = 2$.

Exercise 8D Page 171

5. Integrating Vectors

We can integrate vectors by integrating each function of time separately.

Remember each component will have a constant of integration, $C = (p\boldsymbol{i} + q\boldsymbol{j})$.

Example

A force $\boldsymbol{F}$ acts on a body of mass 250g which is initially at rest at a fixed point O. If $\boldsymbol{F} = ((5t - 2)\boldsymbol{i} + 4t\boldsymbol{j})$N, where t is the time for which the force has been acting on the body, find expressions for:

a) The velocity vector of the body at time t.
b) The position vector of the body at time t.

Example *(Textbook)*

A particle P is moving in a plane so that, at time t seconds, its acceleration is $(4\boldsymbol{i} - 2t\boldsymbol{j})$ms^{-2}. When $t = 3$, the velocity of P is $6\boldsymbol{i}$ ms^{-1} and the position vector of P is $(20\boldsymbol{i} + 3\boldsymbol{j})$ m with respect to a fixed origin O. Find:

(a) the angle between the direction of motion of P and $\boldsymbol{i}$ when $t = 2$

(b) the distance of P from O when $t = 0$.

Test Your Understanding *(EdExcel M2 Jan 2013 Q4)*

At time t seconds the velocity of a particle P is $[(4t-5)\mathbf{i}+3\mathbf{j}]$ m s^{-1}. When $t=0$, the position vector of P is $(2\mathbf{i}+5\mathbf{j})$ m, relative to a fixed origin O.

(a) Find the value of t when the velocity of P is parallel to the vector $\mathbf{j}$.

(1)

(b) Find an expression for the position vector of P at time t seconds.

(4)

A second particle Q moves with constant velocity $(-2\mathbf{i}+c\mathbf{j})$ m s^{-1}. When $t=0$, the position vector of Q is $(11\mathbf{i}+2\mathbf{j})$ m. The particles P and Q collide at the point with position vector $(d\mathbf{i}+14\mathbf{j})$ m.

(c) Find

(i) the value of c,

(ii) the value of d.

(5)

Exercise 8E Page 175
Mixed Exercise 8 Page 177