P2 Chapter 6 :: Trigonometry

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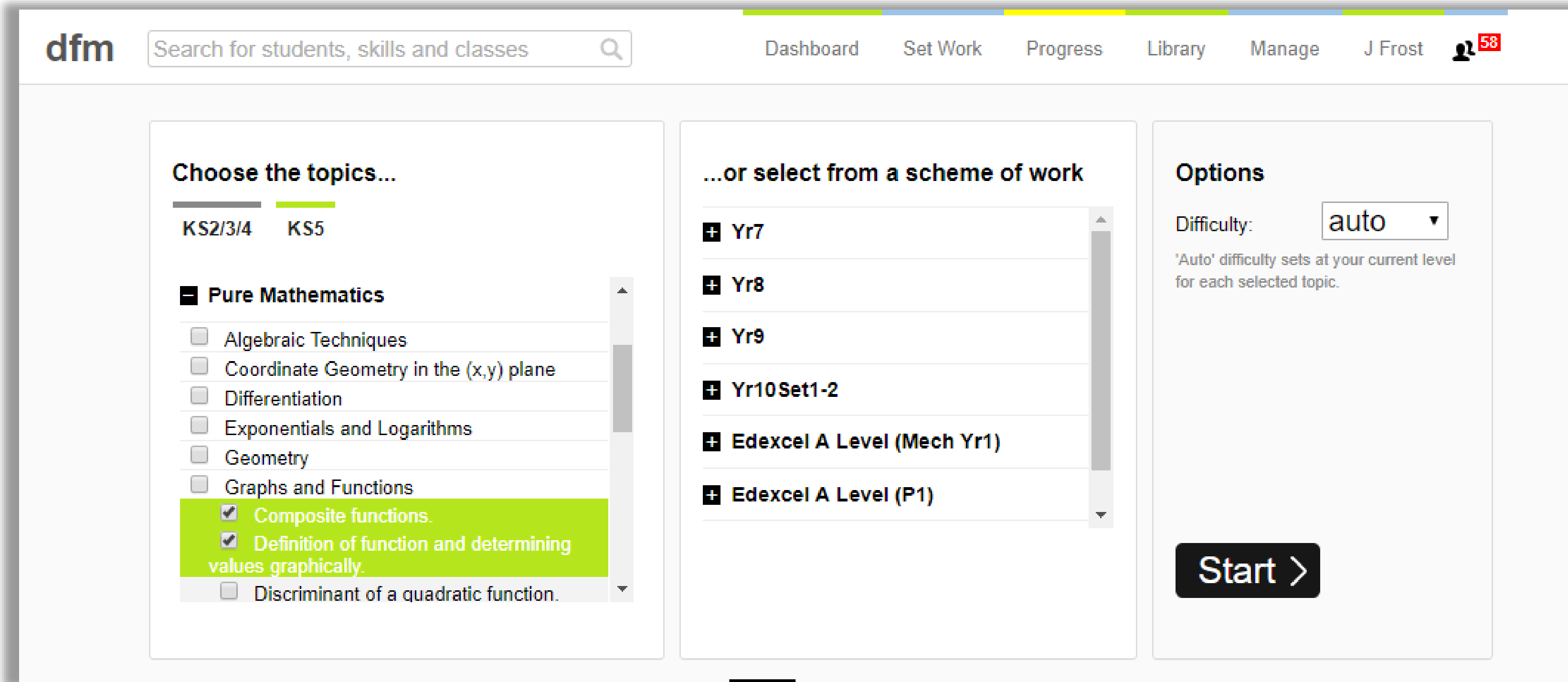

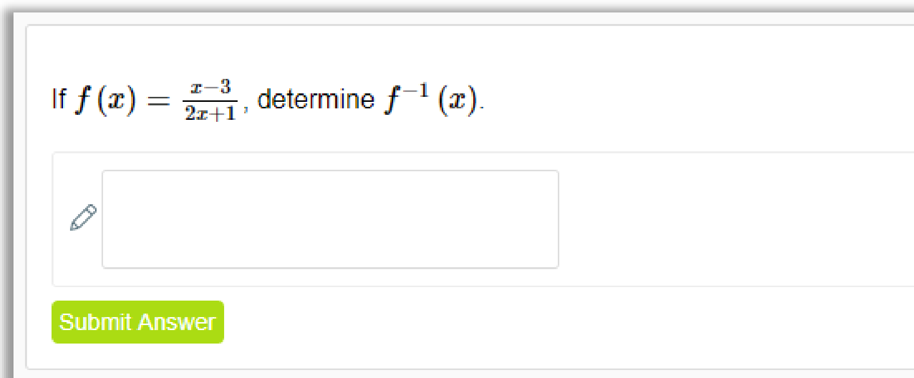

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Chapter Overview

This chapter is very similar to the trigonometry chapters in Year 1. The only difference is that new trig functions: sec, $cosec$ and cot, are introduced.

1:: Understanding $sec, cosec, tan$ and draw their graphs.

"Draw a graph of $y = cosec\ x$ for $0 \leq x < 2\pi$."

2:: 'Solvey' questions.

"Solve, for $0 \leq x < 2\pi$, the equation

$$2cosec^2 x + \cot x = 5$$

giving your solutions to 3sf."

3:: 'Provey' questions.

"Prove that

$$\sec x - \cos x \equiv \sin x \tan x$$"

4:: Inverse trig functions and their domains/ranges.

"Show that, when θ is small, $\sin 5\theta + \tan 2\theta - cos 2\theta \approx 2\theta^2 + 7\theta - 1$."

Teacher Note: There is no change in this chapter relative to the old pre-2017 syllabus.

A new member of the trig family…

$$\cos(x)$$

Original and best. Like the 'Classic Cola' of trig functions*.

$$\cos^2(x) = (\cos x)^2$$

The latter form is particularly useful for differentiation (see Chp9)

$$\cos^{-1}(x) \;\; or \; \arccos(x)$$

Be careful: the -1 here doesn't mean a power of -1 UNLIKE $\cos^2 x$ above. This is an unfortunate historical accident. $arccos$ is an alternative notation we'll see later this chapter.

$$\mathbf{sec}(\boldsymbol{x}) = \frac{\mathbf{1}}{\mathbf{cos}(\boldsymbol{x})}$$

We have a convenient way of representing the reciprocal of the trig functions.

* Actually, I'll contradict this in the 'Just For Your Interest' slides coming up soon.

Reciprocal Trigonometric Functions

$$\sec(x) = \frac{1}{\cos(x)}$$

Short for "**secant**"

$$\operatorname{cosec}(x) = \frac{1}{\sin(x)}$$

Short for "**cosecant**"

$$\cot(x) = \frac{1}{\tan(x)} \quad or \quad \frac{\cos(x)}{\sin(x)}$$

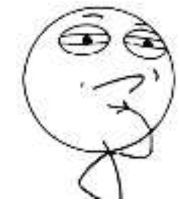
Short for "**cotangent**"

We typically use this version instead of $\frac{1}{tan\,x}$ when doing proof questions.

Tip: To remember these, look at the **3rd letter**: sec's 3rd is 'c' so it's 1 over **cos**.

Just for your interest...

Is 'tangent' (tan) in trigonometry related to the 'tangent' of a circle?

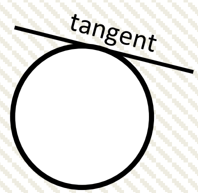


A tangent is a **trigonometric function** (which inputs an angle and gives you the ratio between the opposite and adjacent sides of a right-angled triangle), but **also a line which touches a circle**. Are they related?

?

$$\tan\theta = \frac{opp}{adj}$$

? ?



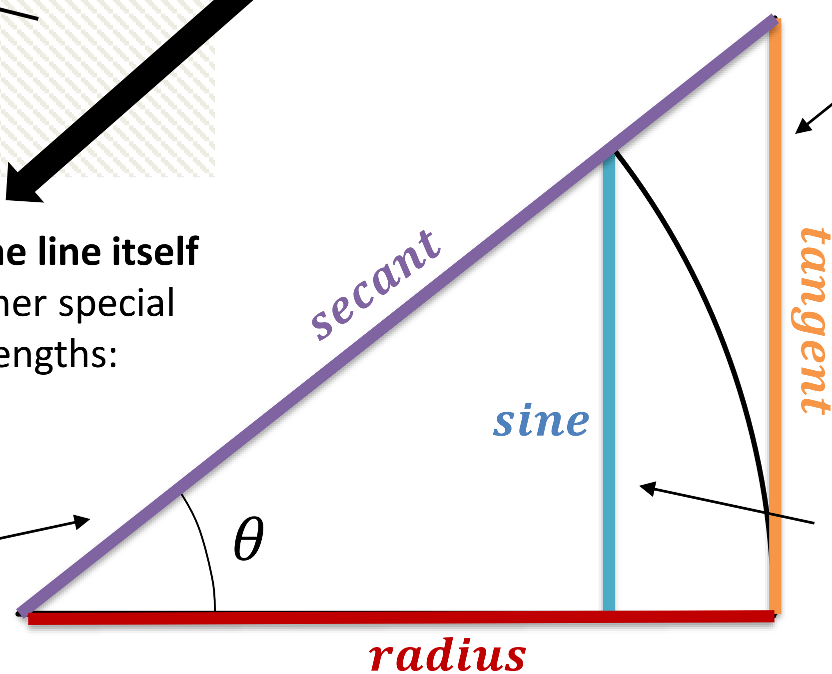
?

A common myth is that sin, cos and tan are the 'core' trigonometric functions.

Actually, they're sin, tan and $\boldsymbol{sec}$!

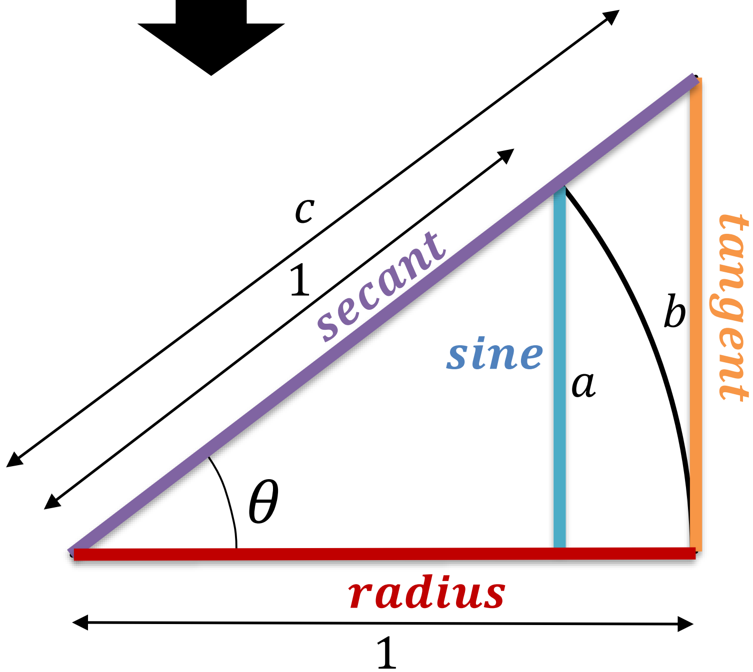
Just as 'radius' can refer either to **the line itself** <u>or</u> its **length**, we have names for other special lines, which can also refer to their lengths:

A **tangent** is *a line which <u>touches</u> the circle*. We're interested in the length just between the touching point and where it meets the secant.



A **secant** (shortened to '*sec*') is a line which <u>*cuts*</u> the circle. In a trig setting, we're interested in the length from the circle centre to where it meets the tangent.

Sine (sort of) comes from the word for 'bowstring'. It refers to half the line if we doubled up the arc and connected the two ends.



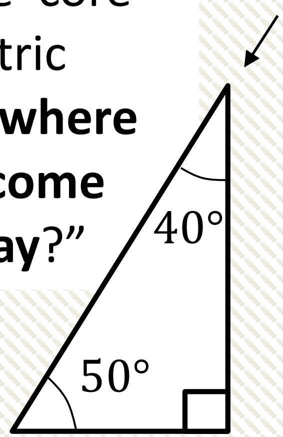
Suppose we let the radius of the sector be 1. Then if we define $\sin\theta = \frac{opp}{hyp}$ and $\tan\theta = \frac{opp}{adj}$ and $\sec\theta = \frac{hyp}{adj}$ then this conveniently gives us:

$$\sin\theta = \frac{a}{1} = a \qquad \tan\theta = \frac{b}{1} = b$$
$$\sec\theta = \frac{c}{1} = c$$

i.e. $\sin\theta$, $\tan\theta$ and $\sec\theta$ actually give us the lengths of the sine, tangent and secant respectively.

And hence $\tan\theta$ and the length of the 'tangent' are clearly linked!

"So if sin, tan and sec are the 'core' trigonometric functions, **where does cos come into the fray**?"



"**cosine**" (cos), "**cotangent**" (cot) and "**cosecant**" ($cosec$) are the '**complementary**' trig functions. Complementary angles add to 90°. The two smaller angles in a right-angled triangle are clearly complementary.

The complementary sine (cosine) of an angle is by definition the sine of the complementary angle, e.g. $\cos(40°) = \sin(50°)$. Thus:

$$\cos(\theta) = \sin(90° - \theta) \qquad \mathrm{cosec}(\theta) = \sec(90° - \theta)$$
$$\cot(\theta) = \tan(90° - \theta)$$

(A very common misconception is that the definition of $\cot\theta \equiv \frac{1}{\tan\theta}$. While this identity is true, this is not the <u>definition</u> of $\cot\theta$, and is a consequence of the adjacent and opposite swapping when we switch to the complementary angle.)

Calculations

You have a calculator in A Level exams, but won't however in STEP, etc. It's good however to know how to calculate certain values yourself if needed.

$\cot\frac{\pi}{4} =$?

$\sec\frac{\pi}{4} =$?

$\operatorname{cosec}\frac{\pi}{3} =$?

$\cot\frac{\pi}{6} =$?

$\operatorname{cosec}\frac{5\pi}{6} =$?

$\cot\frac{\pi}{3} =$?

$\sec\frac{\pi}{6} =$?

$\operatorname{cosec}\frac{\pi}{2} =$?

$\sec\frac{5\pi}{3} =$?

Exercises 6A

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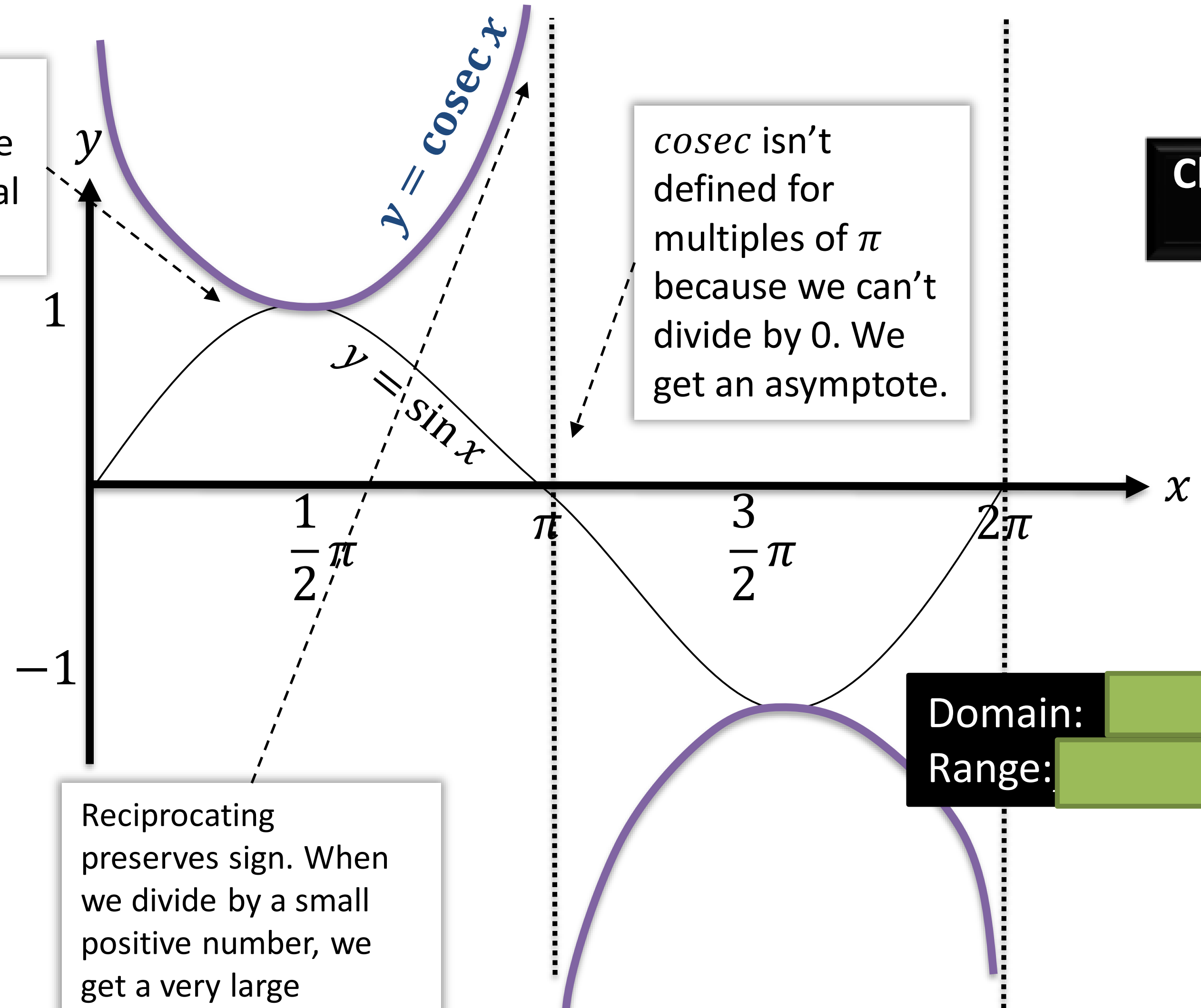
Sketches

To draw a graph of $y = cosec\ x$, start with a graph of $y = \sin x$, then consider what happens when we reciprocate each y value.



It touches here because the reciprocal of 1 is 1.

$cosec$ isn't defined for multiples of π because we can't divide by 0. We get an asymptote.

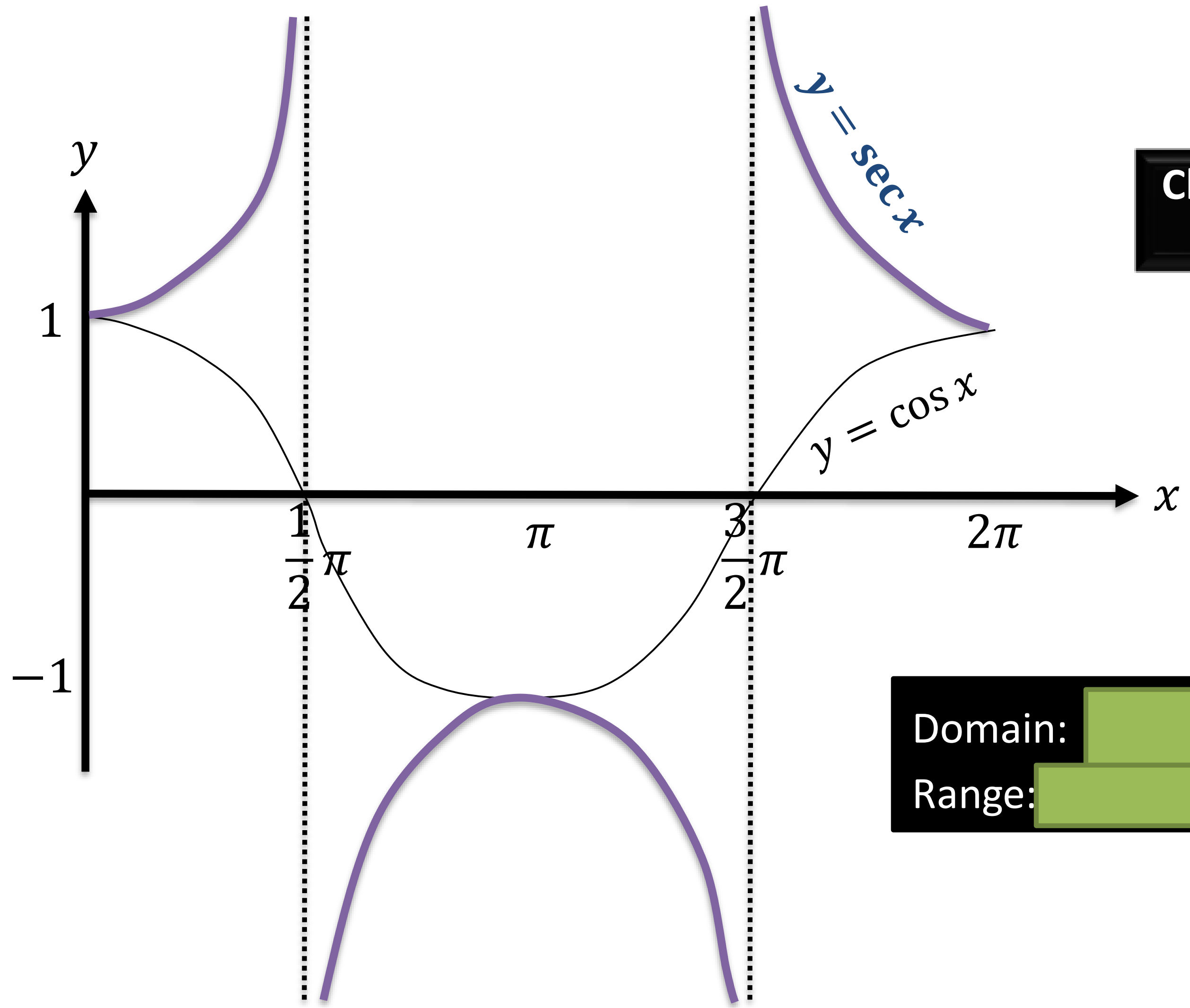
Reciprocating preserves sign. When we divide by a small positive number, we get a very large positive number.

Click to Frosketch $y = cosec\ x$

Domain: ?

Range: ?

Sketches

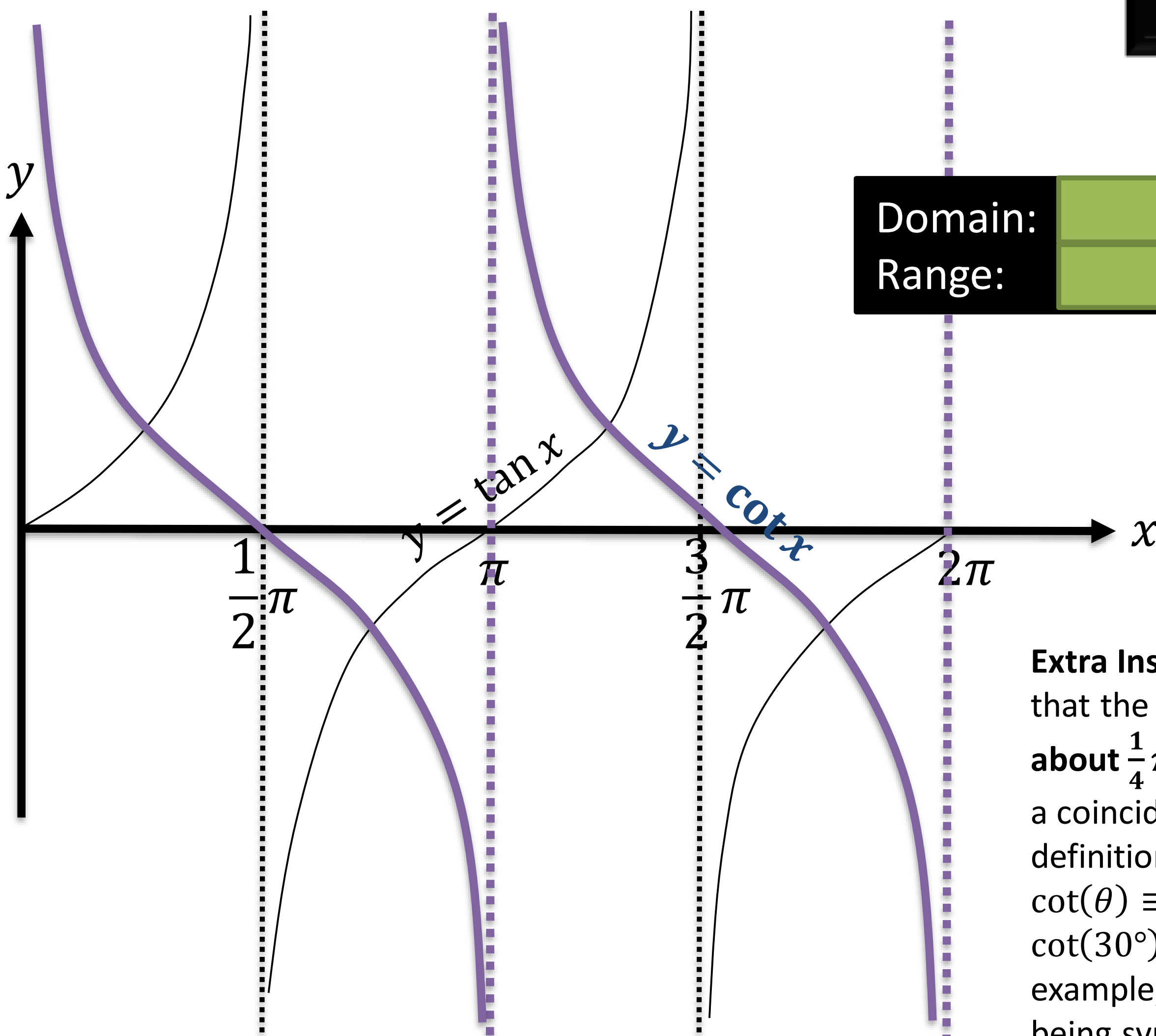


Click to Frosketch
$y = sec\ x$

Domain: ?

Range: ?

Sketches



Click to Brosketch $y = cot\ x$

Domain:	?
Range:	?

Extra Insight: We might spot that the graph is **symmetrical about $\frac{1}{4}\pi, \frac{3}{4}\pi$**, etc. This is not a coincidence: the 'proper' definition of $\cot(\theta) \equiv \tan(90° - \theta)$, so $\cot(30°) = \tan(60°)$ for example, with $30°$ and $60°$ being symmetrical about $45°$.

Example

[Textbook]
a) Sketch the graph of $y = 4cosec\ x, \quad -\pi \leq x \leq \pi$.
b) On the same axes, sketch the line $y = x$.
c) State the number of solutions to the equation $4cosec\ x - x = 0, \ -\pi \leq x \leq \pi$

?

Test Your Understanding

Sketch $y = -1 + \sec 2x$ in the interval $0 \leq x < 360°$.

?

Exercises 6B

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Using $sec, cosec, cot$

Questions in the exam usually come in two flavours: (a) 'provey' questions requiring to prove some identity and (b) 'solvey' questions.

[Textbook]
(a) Simplify $\sin\theta\cot\theta\sec\theta$
(b) Simplify $\sin\theta\cos\theta\ (\sec\theta + cosec\ \theta)$
(c) Prove that $\frac{\cot\theta\ cosec\ \theta}{\sec^2\theta + cosec^2\ \theta} \equiv \cos^3\theta$

Tip 1: Get everything in terms of sin and cos first (using $\cot x = \frac{\cos x}{\sin x}$ rather than $\cot x = \frac{1}{\tan x}$)

Tip 2: Whenever you have algebraic fractions being added/subtracted, whether $\frac{a}{b} + \frac{c}{d}$ or $\frac{a}{b} + c$, combine them into one (as we can typically then use $\sin^2 x + \cos^2 x = 1$)

a

?

b

?

c

?

Test Your Understanding

1 $\sec x - \cos x \equiv \sin x \tan x$

?

2 $(1 + \cos x)(cosec\, x - \cot x) \equiv \sin x$

?

Solvey Questions

[Textbook] Solve the following equations in the interval $0 \leq \theta \leq 360°$:

a) $\sec\theta = -2.5$

b) $\cot 2\theta = 0.6$

a

?

b

?

Solve $\cot\theta = 0$ in the interval $0 \leq \theta \leq 2\pi$.

?

Test Your Understanding

Solve in the interval $0 \leq \theta < 360°$:

$$cosec\ 3\theta = 2$$

?

Exercises 6C

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New Identities

From C2 you knew:

$$\sin^2 x + \cos^2 x = 1$$

There are just two new identities you need to know:

I imagine the Queen coming back from holiday, saying "One is tanned", i.e. the 1 goes with the $\tan^2 x$.

Dividing by $\cos^2 x$:

?

Dividing by $\sin^2 x$:

?

I remember this one by starting with the above, and slapping 'co' on front of each trig function.

"Prove that $1 + \tan^2 x \equiv \sec^2 x$."

$$\sin^2 x + \cos^2 x \equiv 1$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} \equiv \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 \equiv \sec^2 x$$

This has been asked in an exam before! <u>You must explicitly show each term being divided by $\cos^2 x$.</u>

Examples

[Textbook] Prove that $cosec^4\,\theta - \cot^4\theta = \frac{1+\cos^2\theta}{1-\cos^2\theta}$

?

Solve the equation $4\,cosec^2\theta - 9 = \cot\theta$ in the interval $0 \leq \theta \leq 360°$

?

This is just like in AS; if you had say a mixture of $\sin\theta, \sin^2\theta, \cos^2\theta$: you'd change the $\cos^2\theta$ to $1-\sin^2\theta$ in order to get a quadratic in terms of sin.

Test Your Understanding

Edexcel C3 June 2013 (R)

6. (ii) Solve, for $0 \leq \theta < 2\pi$, the equation

$$3\sec^2\theta + 3\sec\theta = 2\tan^2\theta$$

You must show all your working. Give your answers in terms of π.

(6)

?

Q

Solve, for $0 \leq x < 2\pi$, the equation

$$2cosec^2 x + \cot x = 5$$

giving your solutions to 3sf.

?

Exercises 6D

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Inverse Trig Functions

$$If\ sinx = \frac{1}{2}\ then\ x = \sin^{-1}\left(\frac{1}{2}\right)$$

$$We\ also\ call\ this\ \arcsin(\frac{1}{2})\ so\ we\ say\ x = \arcsin(\frac{1}{2})$$

$$\boldsymbol{The\ inverse\ trig\ functions\ are\ known\ as}$$

$$\boldsymbol{y = \arcsin x\ , y = \arccos x, y = \arctan x}$$

They are inverse functions, hence

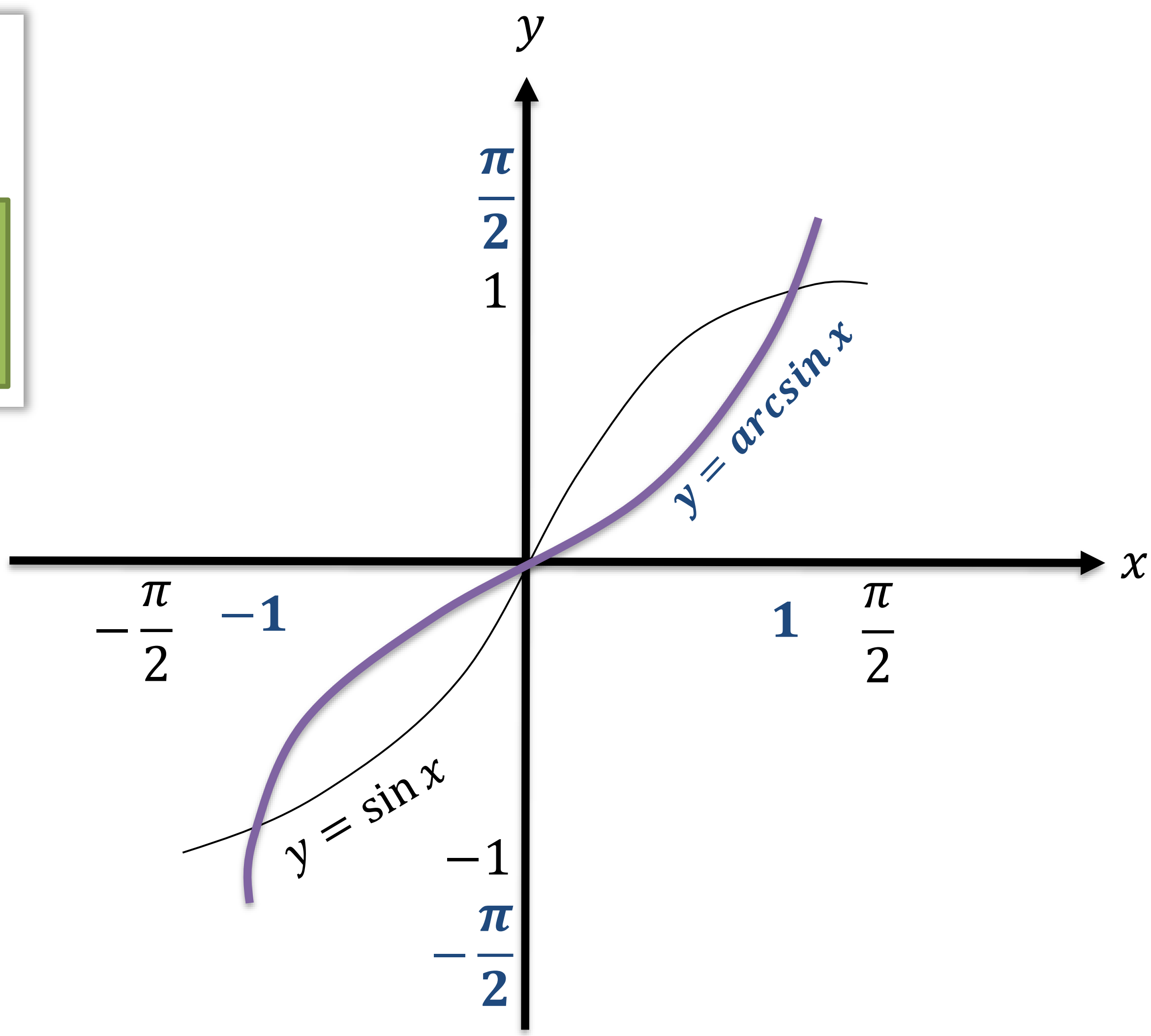
- They only exist for a one to one function
- They map from the range of the original function back to its original domain
- The graphs are reflections of the original in the line y = x.

Inverse Trig Functions

You need to know how to sketch $y = \arcsin x$, $y = \arccos x$, $y = \arctan x$.
(Yes, you could be asked in an exam!)

We have to restrict the domain of $\sin x$ to $-\frac{\pi}{2} \leq x < \frac{\pi}{2}$ before we can find the inverse. Why?

?



Inverse Trig Functions

$y = \arccos x$

?

$y = \arctan x$

?

Evaluating inverse trig functions

[Textbook] Work out, in radians, the values of:

a) $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

b) $\arccos(-1)$

c) $\arctan(\sqrt{3})$

You can simply use the $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$ buttons on your calculator.
If you don't have a calculator, just use the sin, cos, tan graphs backwards.

?

?

?

One Final Problem...

Edexcel C3 Jan 2007

8. (ii) Given that

$$y = \arccos x, \quad -1 \leq x \leq 1 \quad \text{and} \quad 0 \leq y \leq \pi,$$

(*a*) express arcsin x in terms of y.

(2)

(*b*) Hence evaluate arccos x + arcsin x. Give your answer in terms of π.

(1)

Fewer than 10% of candidates got this part right.

?

?

We want to get:
$\sin(f(y)) = x$

Remember:
$\cos(x) = \sin(90 - x)$

Exercises 6E

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