P2 Chapter 10 :: Numerical Methods

jfrost@tiffin.kingston.sch.uk

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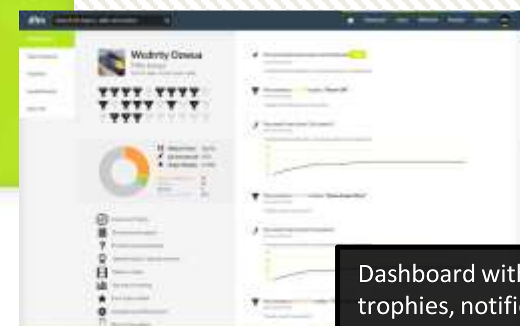

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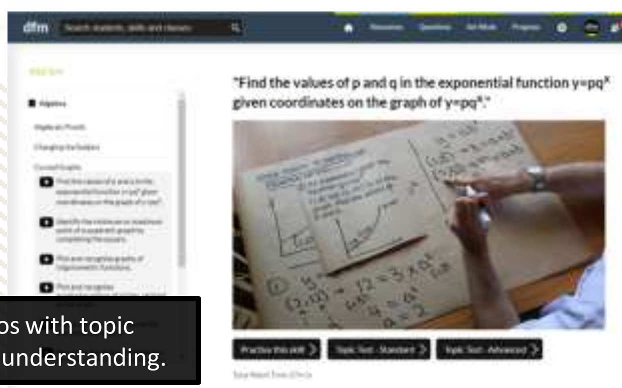

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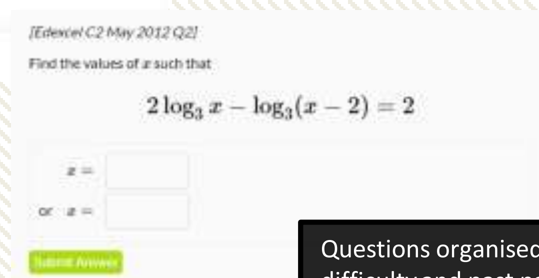


Questions organised by topic, difficulty and past paper.

Chapter Overview

In the GCSE9-1 syllabus you covered 'iteration', which allowed you to find successfully better approximations to the solutions of an equation. We'll revisit this, but also see a more powerful method for approximating solutions.

1:: Locating Roots

What it means to find the root of an equation and when we can be sure a root lies in a stated range.

"Show that $f(x) = x^3 - 4x^2 + 3x + 1$ has a root between $x = 1.4$ and $x = 1.5$".

2:: Using iteration to approximate roots to $f(x) = 0$

[Jan 2010] 2. $\quad \mathrm{f}(x) = x^3 + 2x^2 - 3x - 11$

(*a*) Show that $\mathrm{f}(x) = 0$ can be rearranged as **(2)**

$$x = \sqrt{\left(\frac{3x+11}{x+2}\right)}, \quad x \neq -2.$$

The equation $\mathrm{f}(x) = 0$ has one positive root α.

The iterative formula $x_{n+1} = \sqrt{\left(\frac{3x_n+11}{x_n+2}\right)}$ is used to find an approximation to α.

(*b*) Taking $x_1 = 0$, find, to 3 decimal places, the values of x_2, x_3 and x_4. **(3)**

3:: The Newton-Raphson Method

A numerical method that tends to converge to (i.e. approach) the root faster, by **following the tangent of the graph**.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Notes for teachers: Most of the chapter is covered in GCSE9-1 syllabus (and was in old C3). In addition to this:

- Newton-Raphson method (in old FP1)
- Staircase and cobweb diagrams (in the very old P modules!)
- How to prove a root is correct to a given accuracy (old C3).

Why do we need numerical methods?

Finding the root of a function $f(x)$ is to:

?

Terminology Pedantry: We often say "*Find the roots of* $x^2 + x = 0$", i.e. an **equation** in the form $f(x) = 0$, which really means "*Find the roots of the function* $f(x)$ where $f(x) = x^2 + x$". But we wouldn't say "Find the roots of $x^2 - x = 6$", but "Find the **solutions** of…". So to find roots is to find the x such that the '**output**', **or one side of the equation**, **is 0**.

However, for some functions, the 'exact' root is either complicated and difficult to calculate:

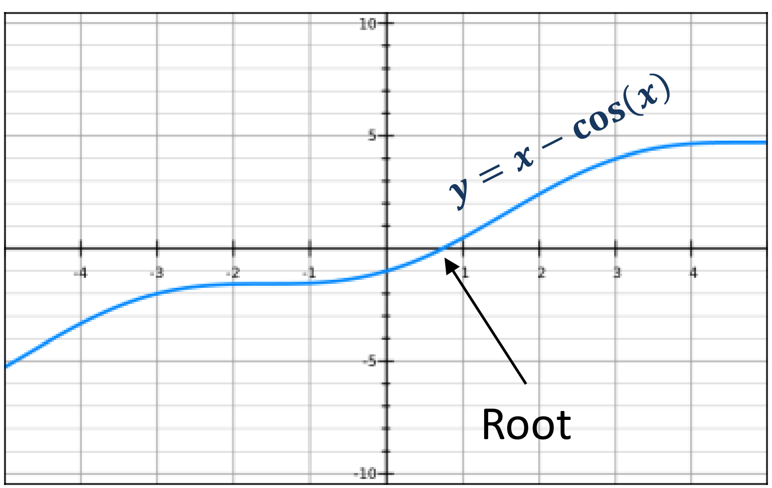
$$x^3 + 2x^2 - 3x + 4 = 0$$

?

or there's no 'algebraic' expression at all! (involving roots, logs, sin, cos, etc.)

$$x - \cos(x) = 0$$

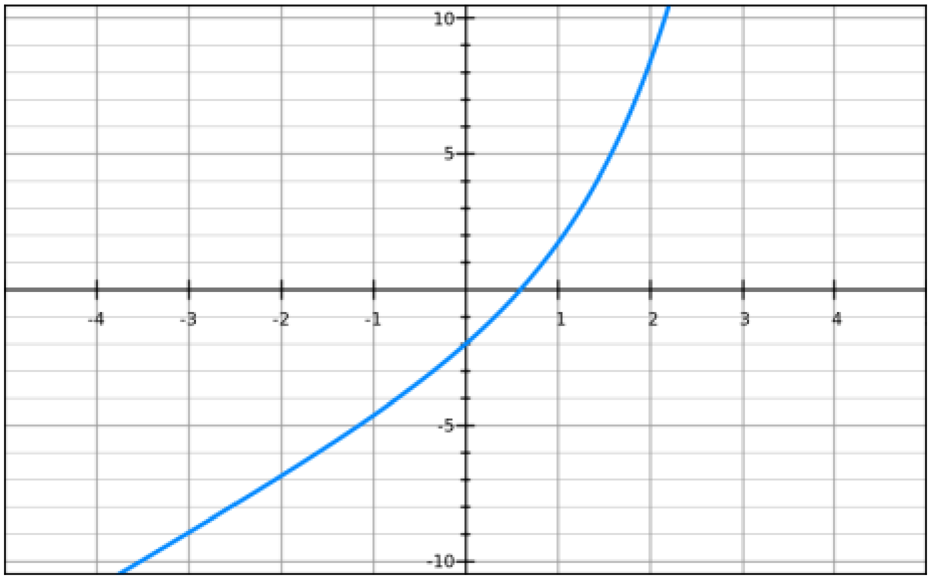
?



But there are a variety of 'numerical methods' which get progressively better solutions to an equation in the form $f(x) = 0$.
You have already seen 'iteration' at GCSE as one such method.

Proving a solution lies in a range

Show that $f(x) = e^x + 2x - 3$ has a root between $x = 0.5$ and $x = 0.6$



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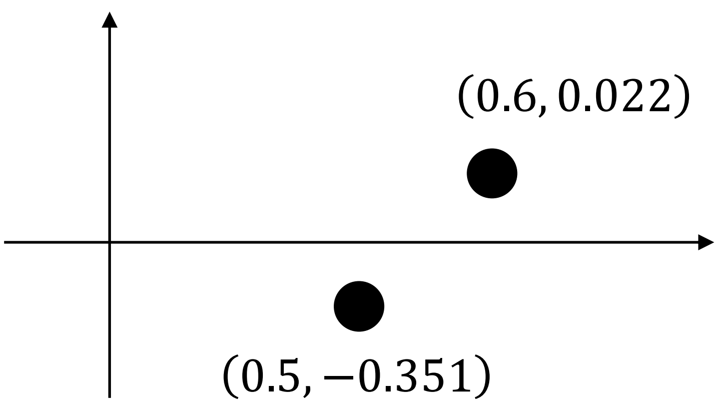


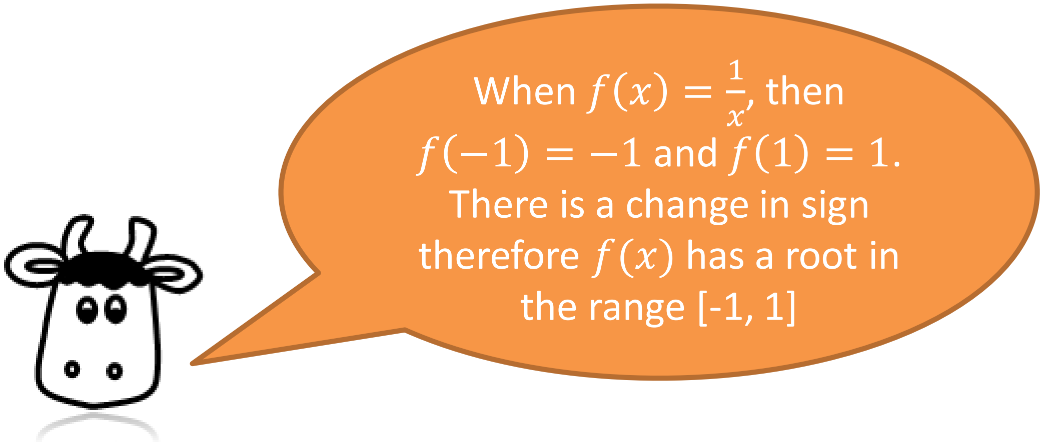
If the y value goes from negative to positive or vice versa, then clearly the y values must pass 0 somewhere in between.

Bro Exam Tip: In the mark scheme they're looking for:

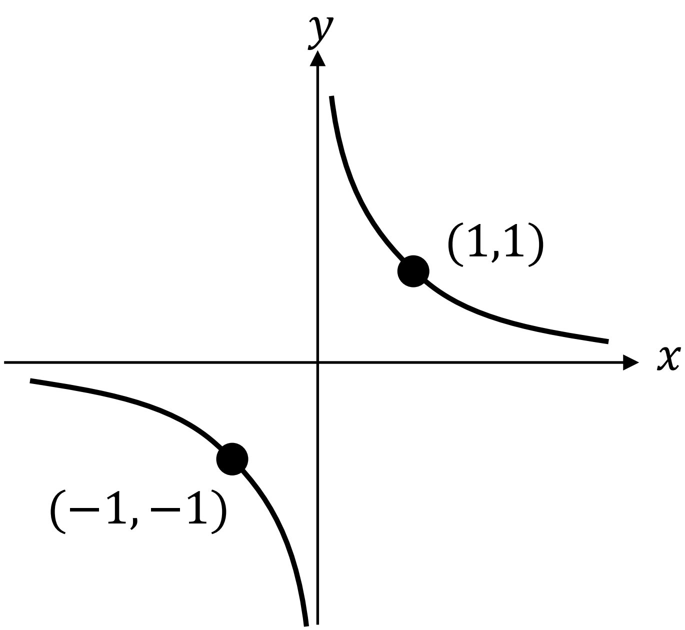
1. Finding the function output for the two values.
2. Referring to a 'change in sign'.

…but only if the function is continuous

Stupid Steve says:

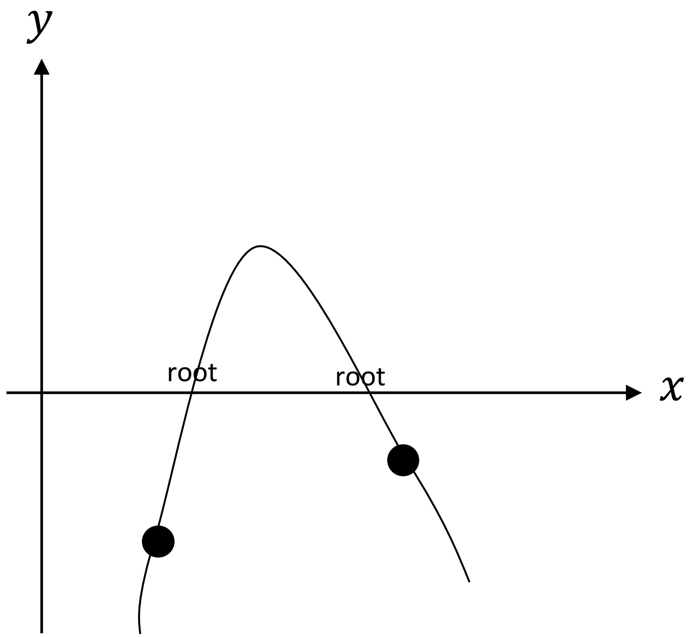


Why is Steve wrong?



A function is **continuous** if the line **does not 'jump'**. A root is only guaranteed with a sign change if the function is continuous, as otherwise the line can skip past 0 (in this case due to a vertical asymptote.

…and no sign change doesn't mean there isn't a root



Beware! Just because there isn't a sign change, doesn't mean there's no root in that interval.

The sign change method fails to detect a root if there were an **even number of roots** in that interval.

Proving a solution to a given accuracy

Edexcel C3 Jan 2013

$$g(x) = e^{x-1} + x - 6$$

The root of $g(x) = 0$ is α.

(c) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places.

(3)

Thinking back to lower school, if the root 2.307 is correct to 3dp, what's the smallest and greatest value it could be?

Smallest: ?

Greatest: ?

If there was a sign change between $x = 2.3065$ and $x = 2.3075$, then we know $2.3065 < \alpha < 2.3075$. But if the value is in this range, then we know it is 2.307 to 3 decimal places!

? What do we write in exam…

Example

[Textbook] (a) Using the same axes, sketch the graphs of $y = \ln x$ and $y = \frac{1}{x}$. Explain how your diagrams shows that the function $y = \ln(x) - \frac{1}{x}$ has only one root.
(b) Show that this root lies in the interval $1.7 < x < 1.8$
(c) Given that the root of $f(x)$ is α, show that $\alpha = 1.763$ correct to 3 decimal places.

a

?

b

?

c

?

Exercise 10A

Pearson Pure Mathematics Year 2/AS
Pages 276-277

(Classes pressed for time may wish to skip this exercise)

Using iteration to approximate a root

Edexcel C3 Jan 2013

$$g(x) = e^{x-1} + x - 6$$

(*a*) Show that the equation $g(x) = 0$ can be written as

$$x = \ln(6 - x) + 1, \qquad x < 6.$$

(2)

The root of $g(x) = 0$ is α.

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \qquad x_0 = 2.$$

is used to find an approximate value for α.

(*b*) Calculate the values of x_1, x_2 and x_3 to 4 decimal places.

(3)

(*c*) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places.

(3)

a

?

Fro Tip: The difficulty is that there's multiple choices of x to isolate on one side of the equation. Therefore use the target equation to give clues for how to rearrange.

✐ To solve $f(x) = 0$ by an iterative method, rearrange into a form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$

We'll see why it works later.

Using iteration to approximate a root

Edexcel C3 Jan 2013

$$g(x) = e^{x-1} + x - 6$$

(*a*) Show that the equation $g(x) = 0$ can be written as

$$x = \ln(6 - x) + 1, \qquad x < 6.$$

(2)

The root of $g(x) = 0$ is α.

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \qquad x_0 = 2.$$

is used to find an approximate value for α.

(*b*) Calculate the values of x_1, x_2 and x_3 to 4 decimal places.

(3)

(*c*) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places.

(3)

<u>x_0, x_1, x_2 represent successively better approximations of the root</u>, where x_0 is the starting value.

Froculator Tip: Initially type x_0 (i.e. 2) onto your calculator.
Now just type:

$$\ln(6 - ANS) + 1$$

And then spam your $=$ key to get successive iterations.

b

?

Exam Tip: Show the substitution for x_1 to ensure you get the method mark. But then just write the final value for x_2 and thereafter, as the remaining marks will be 'accuracy' ones.

Does the starting value x_0 matter?

Yes! We'll see why in a bit when we look at staircase and cobweb diagrams.

- If there are a multiple roots, iteration might converge to (i.e. approach) a different root.
- Or we may not converge to a root at all, and **diverge** (i.e. approach infinity).

[Textbook] $f(x) = x^3 - 3x^2 - 2x + 5$

(a) Show that the equation $f(x) = 0$ has a root in the interval $3 < x < 4$.

(b) Use the iterative formula $x_{n+1} = \sqrt{\frac{x_n^3 - 2x_n + 5}{3}}$ to calculate the values of x_1, x_2 and x_3, giving your answers to 4 decimal places, and taking:
(i) $x_0 = 1.5$ (ii) $x_0 = 4$

(a) ?

(b) (ii) ?

(b) i) ?

Test Your Understanding

Edexcel C3 June 2012 Q2

$$f(x) = x^3 + 3x^2 + 4x - 12$$

(*a*) Show that the equation $f(x) = 0$ can be written as

$$x = \sqrt{\left(\frac{4(3-x)}{(3+x)}\right)}, \quad x \neq -3. \qquad \textbf{(3)}$$

The equation $x^3 + 3x^2 + 4x - 12 = 0$ has a single root which is between 1 and 2.

(*b*) Use the iteration formula

$$x_{n+1} = \sqrt{\left(\frac{4(3-x_n)}{(3+x_n)}\right)}, \quad n \geq 0,$$

with $x_0 = 1$ to find, to 2 decimal places, the value of x_1, x_2 and x_3. **(3)**

The root of $f(x) = 0$ is α.

(*c*) By choosing a suitable interval, prove that $\alpha = 1.272$ to 3 decimal places. **(3)**

Warning: Any particular mark scheme gives what is <u>minimally</u> acceptable. So you should use the full wording earlier to avoid the risk of mark loss.

(a)

?

(b)

?

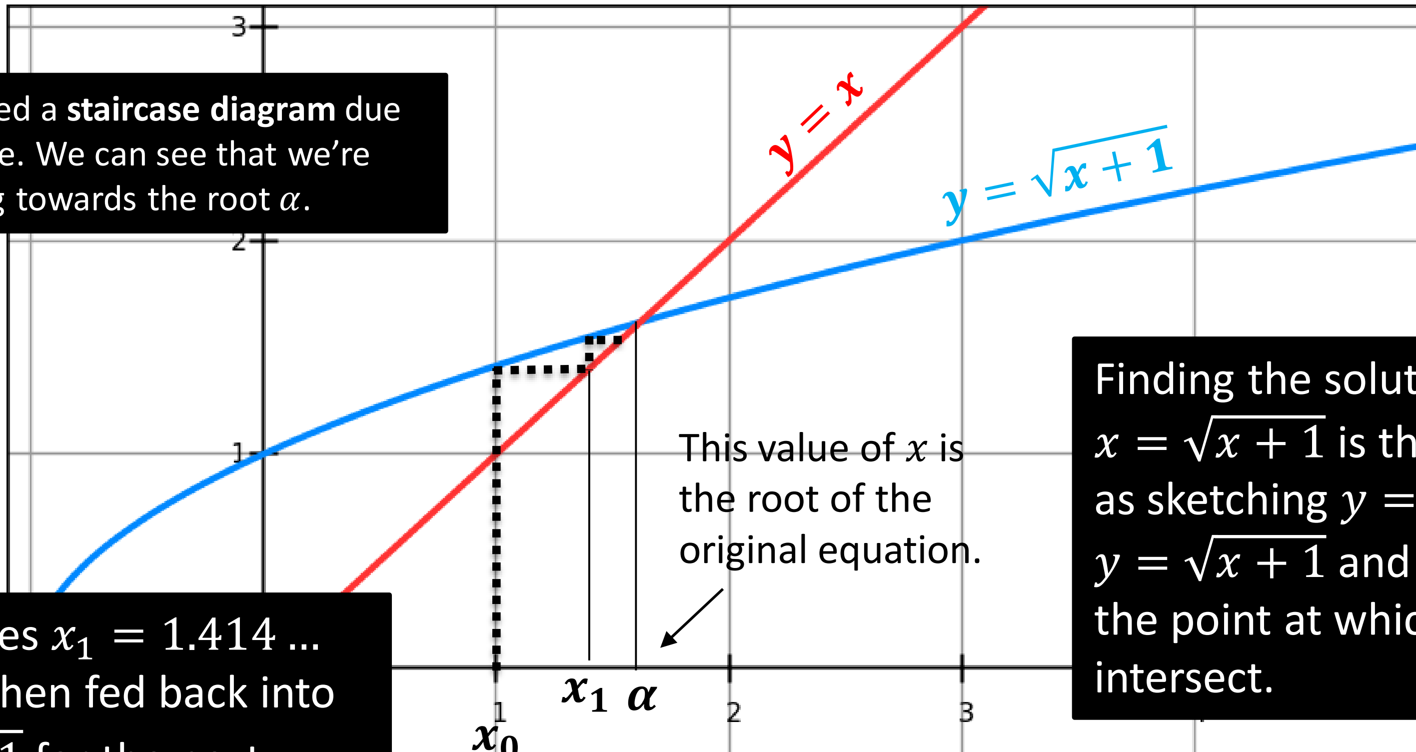
(c)

?

Why does this method work?

Solve $x^2 - x - 1 = 0$

Recall we put in the form $x = g(x)$: in this case $x = \sqrt{x+1}$ is one possible rearrangement.
We can then use the recurrence $x_{n+1} = \sqrt{x_n + 1}$. Why does this recurrence work?



This is called a **staircase diagram** due to its shape. We can see that we're converging towards the root α.

Finding the solution to $x = \sqrt{x+1}$ is the same as sketching $y = x$ and $y = \sqrt{x+1}$ and seeing the point at which they intersect.

When $x_0 = 1$, we would find $\sqrt{x_0 + 1}$. This is the y value on the $y = \sqrt{x+1}$ graph.

This gives $x_1 = 1.414 \ldots$ This is then fed back into $\sqrt{x_n + 1}$ for the next iteration, i.e. the y value becomes the new x value! This is equivalent to moving to the line $y = x$.

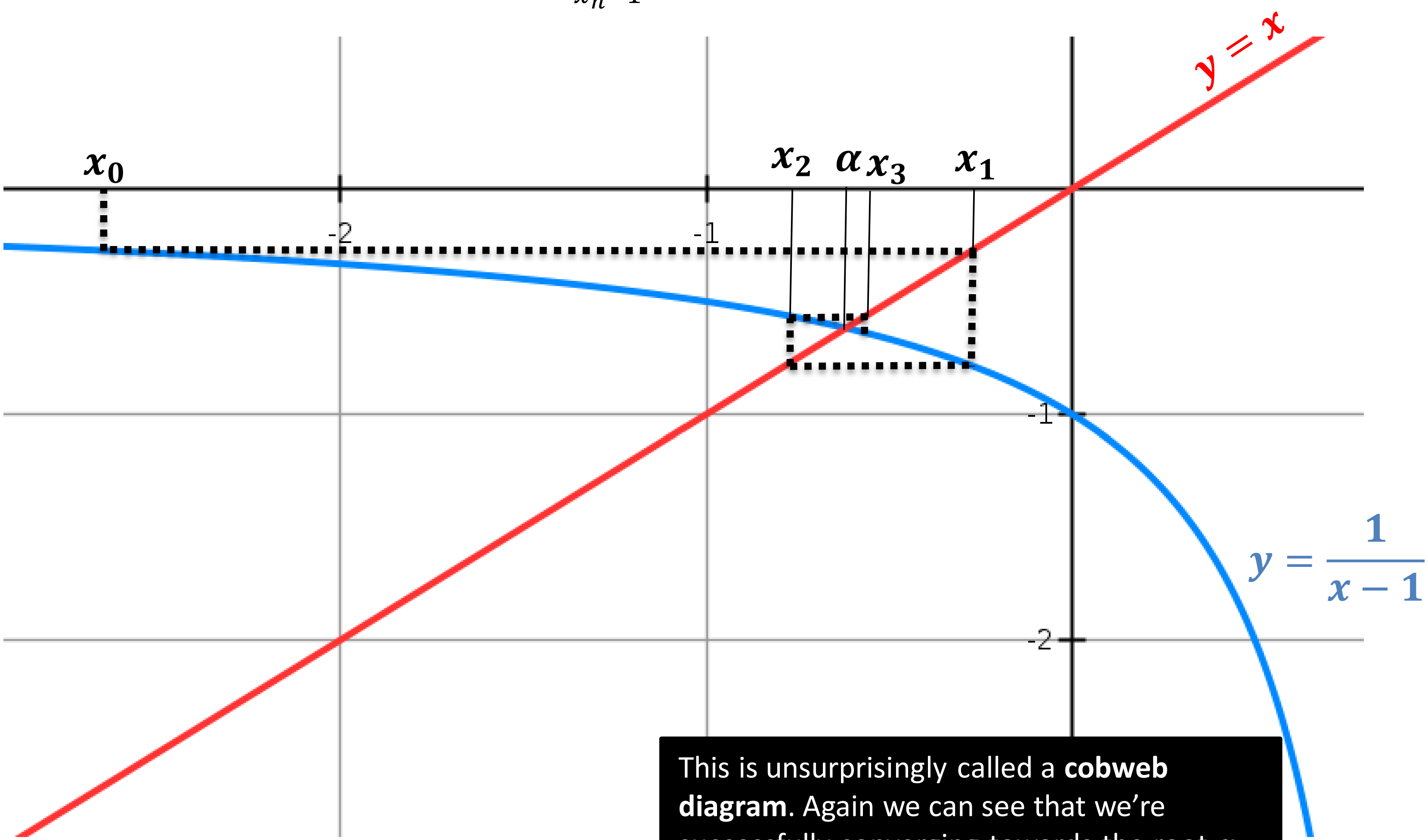
We can repeat this process using $x_1 = 1.414 \ldots$ to get x_2 and so on.

Cobweb Diagrams

Solve $x^2 - x - 1 = 0$

We could also have rearranged differently to $x = \frac{1}{x-1}$

Therefore we use the recurrence $x_{n+1} = \frac{1}{x_n - 1}$. What happens this time?



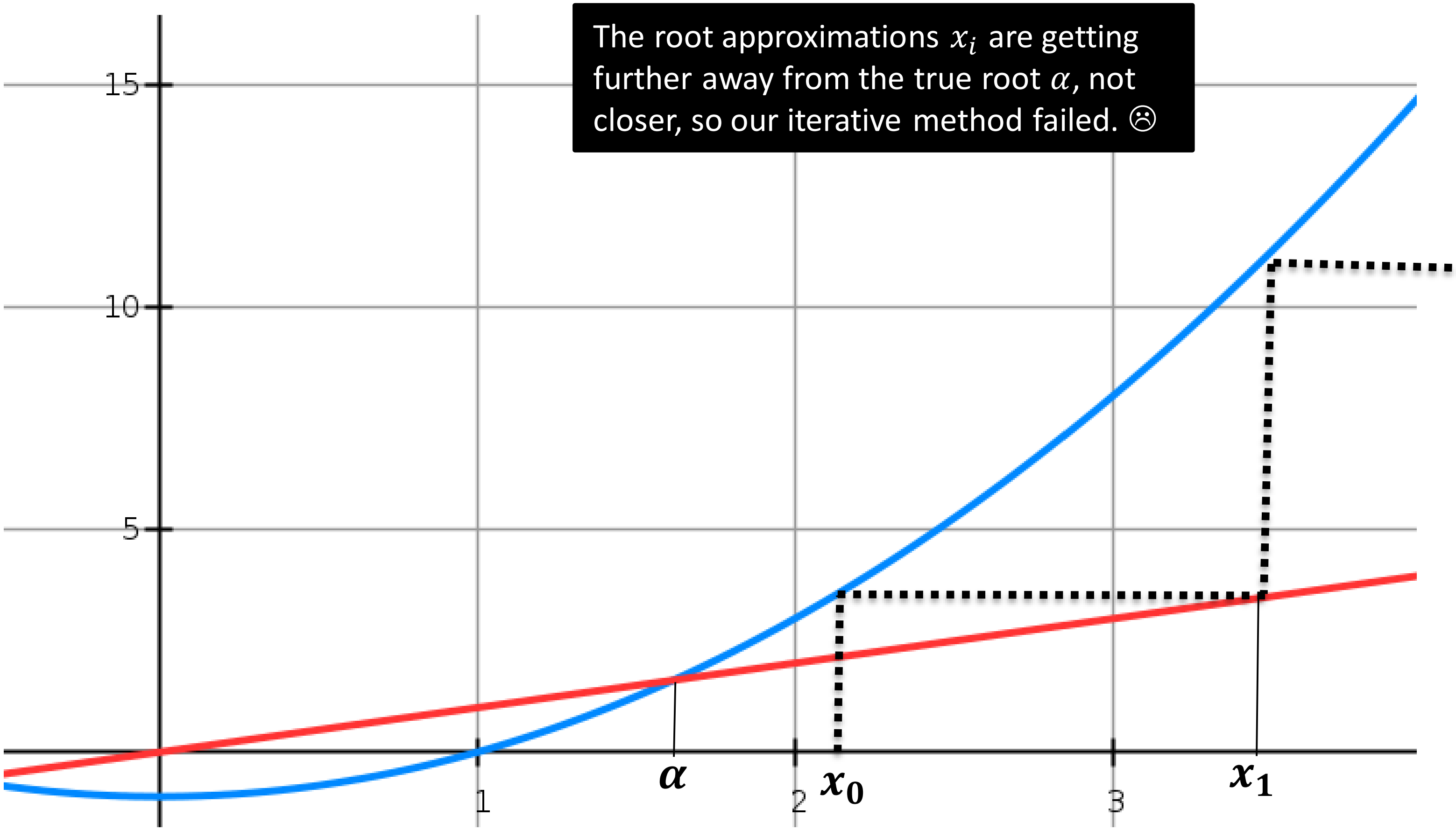
This is unsurprisingly called a **cobweb diagram**. Again we can see that we're successfully converging towards the root α.

And when iteration fails...

Solve $x^2 - x - 1 = 0$

But again, we could have rearranged differently! $x = x^2 - 1$
Therefore we use the recurrence $x_{n+1} = x_n^2 - 1$. What happens this time?

The root approximations x_i are getting further away from the true root α, not closer, so our iterative method failed. ☹



Test Your Understanding

$$f(x) = x^2 - 8x + 4$$

(a) Show that the root of the equation $f(x) = 0$ can be written as $x = \sqrt{8x - 4}$

(b) Using the iterative formula $x_{n+1} = \sqrt{8x_n - 4}$, and starting with $x_0 = 1$, draw a staircase diagram, indicating x_0, x_1, x_2 on your x-axis, as well as the root α.

?

Exercise 10B

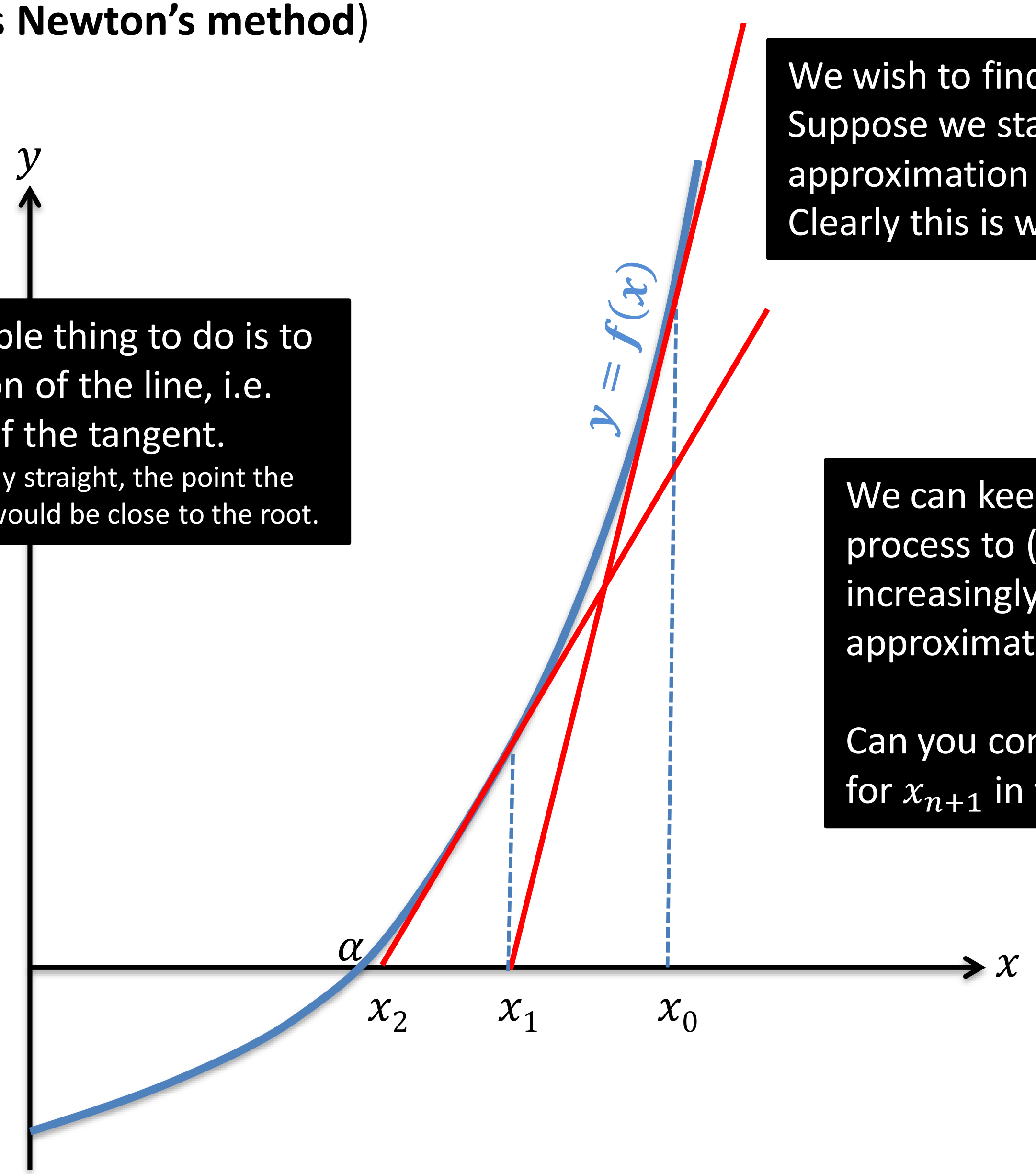
Pearson Pure Mathematics Year 2/AS
Pages 280-282

The Newton-Raphson Process

(also known as **Newton's method**)

We wish to find the root α.
Suppose we start with the indicated approximation of the root, x_0.
Clearly this is well off the mark!

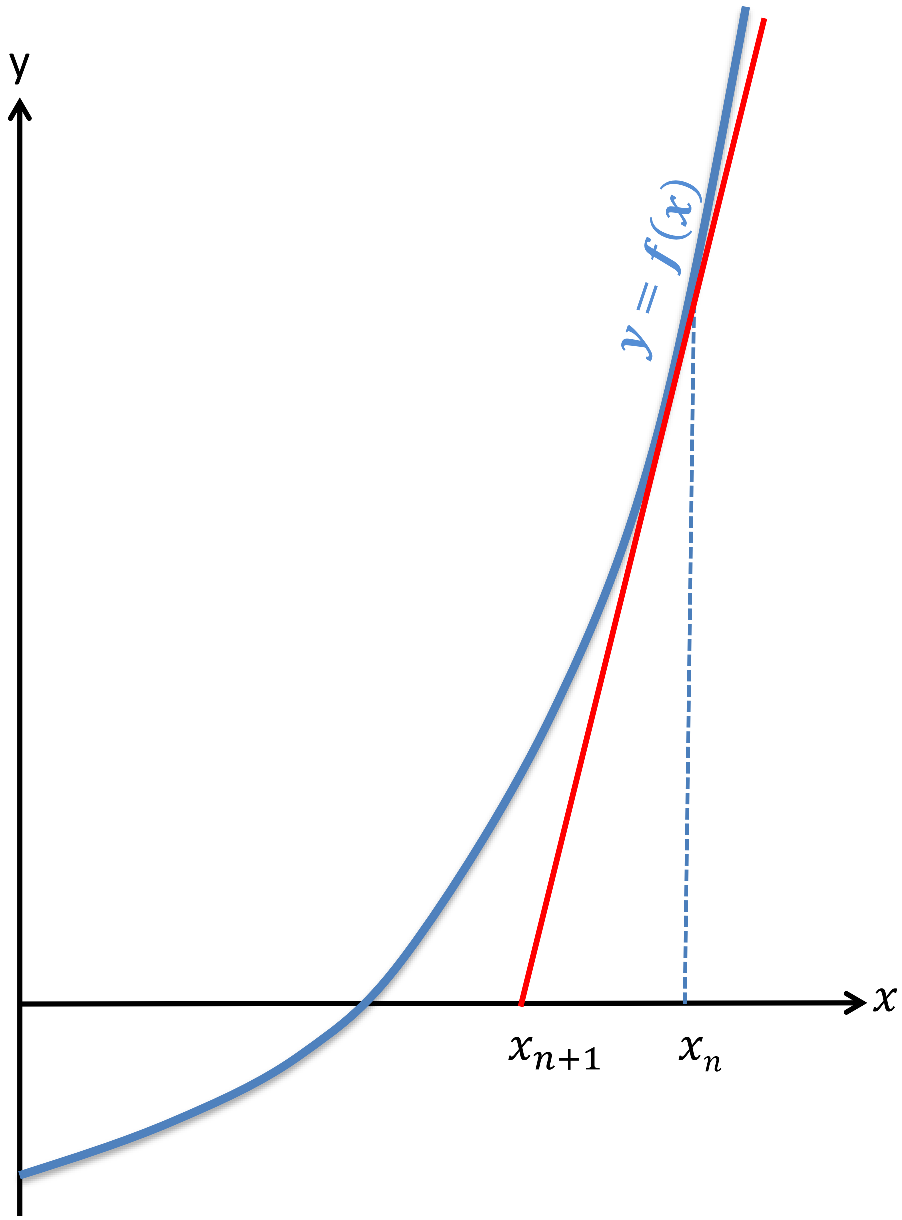
A seemingly sensible thing to do is to follow the direction of the line, i.e. use the gradient of the tangent.
If the line was reasonably straight, the point the tangent hits the x-axis would be close to the root.



We can keep repeating this process to (hopefully) get increasingly accurate approximations.

Can you come up with a formula for x_{n+1} in terms of x_n?

The Newton-Raphson Process

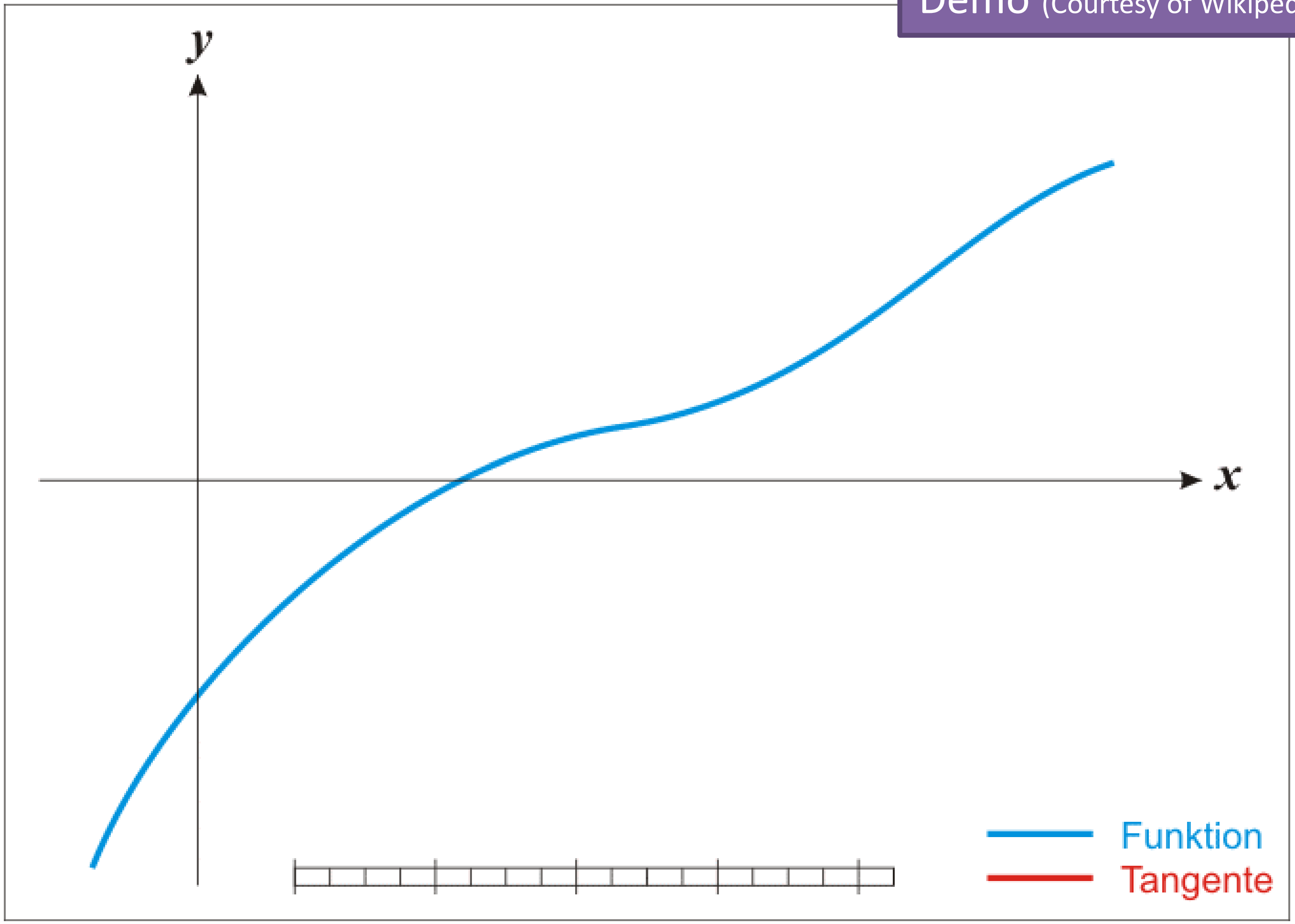


Formula:

?

A nice animation…

Demo (Courtesy of Wikipedia)



This is German, but I’ll presume you’re not an idiot.

Example

Returning to our original example: $\boldsymbol{x = \cos(x)}$, say letting $x_0 = 0.5$

(Note: Recall that differentiation assumes radians)

?

r

Quickfire Questions

Using the Newton-Raphson process, state the recurrence relation for the following functions:

$f(x) = x^3 - 2$	?
$f(x) = \tan x$	?
$f(x) = x^2 - x - 1$	?

Example Exam Question

Edexcel FP1 June 2013(R) Q3c

$$f(x) = \frac{1}{2}x^4 - x^3 + x - 3$$

...

The equation $f(x) = 0$ has a root β in the interval $[-2, -1]$.

(*c*) Taking -1.5 as a first approximation to β, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to β.
Give your answer to 2 decimal places.

(5)

?

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Test Your Understanding

Edexcel FP1 Jan 2010 Q2c

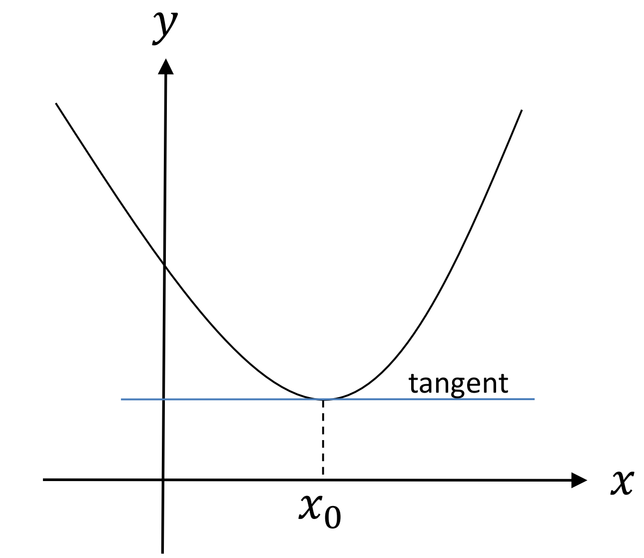
$$f(x) = 3x^2 - \frac{11}{x^2}.$$

(*c*) Taking 1.4 as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to obtain a second approximation to α, giving your answer to 3 decimal places.

(5)

?

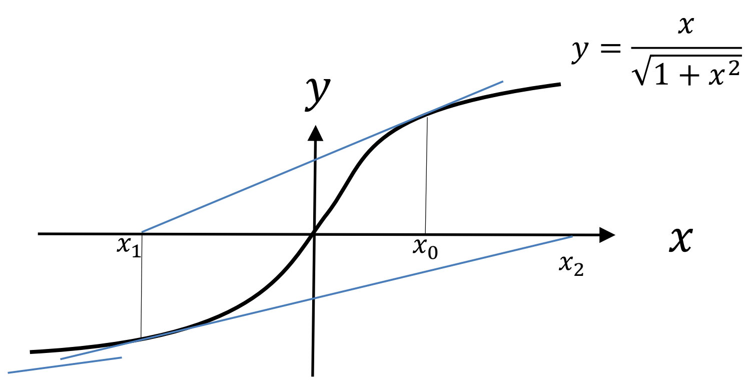
When does Newton-Raphson fail?



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If the starting value x_0 was the stationary point, then $f'(x_0) = 0$, resulting in a division by 0 in the above formula.

Graphically, it is because the tangent will never reach the x-axis.



Newton-Raphson also suffers from the same drawbacks as solving by iteration, in that it's possible for the values of x_i to **diverge**.

In this example, the x_i oscillate either side of 0, but gradually getting further away from $\alpha = 0$.

Exercise 10C

Pearson Pure Mathematics Year 2/AS
Pages 284-285

Application to Modelling

[Textbook] The price of a car in £s, x years after purchase, is modelled by the function

$$f(x) = 15\,000\,(0.85)^x - 1000\sin x\,, \qquad x > 0$$

(a) Use the model to find the value, to the nearest hundred £s, of the car 10 years after purchase.

(b) Show that $f(x)$ has a root between 19 and 20.

(c) Find $f'(x)$

(d) Taking 19.5 as a first approximation, apply the Newton-Raphson method once to $f(x)$ to obtain a second approximation for the time when the value of the car is zero. Give your answer to 3 decimal places.

(e) Criticise this model with respect to the value of the car as it gets older.

a

?

b

?

c

?

d

?

Exercise 10D

Pearson Pure Mathematics Year 2/AS
Pages 287-289