# Solving quadratic equations by factorisation

**A LEVEL LINKS**
**Scheme of work:** 1b. Quadratic functions – factorising, solving, graphs and the discriminants

## Key points

- A quadratic equation is an equation in the form $ax^2 + bx + c = 0$ where $a \neq 0$.
- To factorise a quadratic equation find two numbers whose sum is $b$ and whose products is $ac$.
- When the product of two numbers is 0, then at least one of the numbers must be 0.
- If a quadratic can be solved it will have two solutions (these may be equal).

## Examples

**Example 1** Solve $5x^2 = 15x$

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|---|---|
| $5x^2 = 15x$ | **1** Rearrange the equation so that all of the terms are on one side of the equation and it is equal to zero. Do not divide both sides by $x$ as this would lose the solution $x = 0$. |
| $5x^2 - 15x = 0$ | |
| $5x(x - 3) = 0$ | **2** Factorise the quadratic equation. $5x$ is a common factor. |
| So $5x = 0$ or $(x - 3) = 0$ | **3** When two values multiply to make zero, at least one of the values must be zero. |
| Therefore $x = 0$ or $x = 3$ | **4** Solve these two equations. |

**Example 2** Solve $x^2 + 7x + 12 = 0$

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|---|---|
| $x^2 + 7x + 12 = 0$ | **1** Factorise the quadratic equation. Work out the two factors of $ac = 12$ which add to give you $b = 7$. (4 and 3) |
| $b = 7$, $ac = 12$ | |
| $x^2 + 4x + 3x + 12 = 0$ | **2** Rewrite the $b$ term ($7x$) using these two factors. |
| $x(x + 4) + 3(x + 4) = 0$ | **3** Factorise the first two terms and the last two terms. |
| $(x + 4)(x + 3) = 0$ | **4** $(x + 4)$ is a factor of both terms. |
| So $(x + 4) = 0$ or $(x + 3) = 0$ | **5** When two values multiply to make zero, at least one of the values must be zero. |
| Therefore $x = -4$ or $x = -3$ | **6** Solve these two equations. |

**Example 3** Solve $9x^2 - 16 = 0$

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|---|---|
| $9x^2 - 16 = 0$<br>$(3x + 4)(3x - 4) = 0$<br><br>So $(3x + 4) = 0$ or $(3x - 4) = 0$<br><br>$x = -\frac{4}{3}$ or $x = \frac{4}{3}$ | **1** Factorise the quadratic equation. This is the difference of two squares as the two terms are $(3x)^2$ and $(4)^2$.<br>**2** When two values multiply to make zero, at least one of the values must be zero.<br>**3** Solve these two equations. |

**Example 4** Solve $2x^2 - 5x - 12 = 0$

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|---|---|
| $b = -5$, $ac = -24$<br><br><br><br>So $2x^2 - 8x + 3x - 12 = 0$<br><br>$2x(x - 4) + 3(x - 4) = 0$<br><br>$(x - 4)(2x + 3) = 0$<br>So $(x - 4) = 0$ or $(2x + 3) = 0$<br><br>$x = 4$ or $x = -\frac{3}{2}$ | **1** Factorise the quadratic equation. Work out the two factors of $ac = -24$ which add to give you $b = -5$. ($-8$ and 3)<br>**2** Rewrite the $b$ term ($-5x$) using these two factors.<br>**3** Factorise the first two terms and the last two terms.<br>**4** $(x - 4)$ is a factor of both terms.<br>**5** When two values multiply to make zero, at least one of the values must be zero.<br>**6** Solve these two equations. |

# Practice

**1** Solve

**a** $6x^2 + 4x = 0$
**b** $28x^2 - 21x = 0$
**c** $x^2 + 7x + 10 = 0$
**d** $x^2 - 5x + 6 = 0$
**e** $x^2 - 3x - 4 = 0$
**f** $x^2 + 3x - 10 = 0$
**g** $x^2 - 10x + 24 = 0$
**h** $x^2 - 36 = 0$
**i** $x^2 + 3x - 28 = 0$
**j** $x^2 - 6x + 9 = 0$
**k** $2x^2 - 7x - 4 = 0$
**l** $3x^2 - 13x - 10 = 0$

**2** Solve

**a** $x^2 - 3x = 10$
**b** $x^2 - 3 = 2x$
**c** $x^2 + 5x = 24$
**d** $x^2 - 42 = x$
**e** $x(x + 2) = 2x + 25$
**f** $x^2 - 30 = 3x - 2$
**g** $x(3x + 1) = x^2 + 15$
**h** $3x(x - 1) = 2(x + 1)$

**Hint**

Get all terms onto one side of the equation.

# Solving quadratic equations by completing the square

**A LEVEL LINKS**
**Scheme of work:** 1b. Quadratic functions – factorising, solving, graphs and the discriminants

## Key points

- Completing the square lets you write a quadratic equation in the form $p(x + q)^2 + r = 0$.

## Examples

**Example 5** Solve $x^2 + 6x + 4 = 0$. Give your solutions in surd form.

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|---|---|
| $x^2 + 6x + 4 = 0$ | **1** Write $x^2 + bx + c = 0$ in the form $\left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = 0$ |
| $(x + 3)^2 - 9 + 4 = 0$ | **2** Simplify. |
| $(x + 3)^2 - 5 = 0$<br>$(x + 3)^2 = 5$ | **3** Rearrange the equation to work out $x$. First, add 5 to both sides. |
| $x + 3 = \pm\sqrt{5}$ | **4** Square root both sides. Remember that the square root of a value gives two answers. |
| $x = \pm\sqrt{5} - 3$ | **5** Subtract 3 from both sides to solve the equation. |
| So $x = -\sqrt{5} - 3$ or $x = \sqrt{5} - 3$ | **6** Write down both solutions. |

**Example 6** Solve $2x^2 - 7x + 4 = 0$. Give your solutions in surd form.

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|---|---|
| $2x^2 - 7x + 4 = 0$ | **1** Before completing the square write $ax^2 + bx + c$ in the form $a\left(x^2 + \frac{b}{a}x\right) + c$ |
| $2\left(x^2 - \frac{7}{2}x\right) + 4 = 0$ | **2** Now complete the square by writing $x^2 - \frac{7}{2}x$ in the form $\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2$ |
| $2\left[\left(x - \frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2\right] + 4 = 0$ | |
| $2\left(x - \frac{7}{4}\right)^2 - \frac{49}{8} + 4 = 0$ | **3** Expand the square brackets. |
| $2\left(x - \frac{7}{4}\right)^2 - \frac{17}{8} = 0$ | **4** Simplify. |

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| | |
|---|---|
| $2\left(x-\frac{7}{4}\right)^2=\frac{17}{8}$ | **5** Rearrange the equation to work out $x$. First, add $\frac{17}{8}$ to both sides. |
| $\left(x-\frac{7}{4}\right)^2=\frac{17}{16}$ | **6** Divide both sides by 2. |
| $x-\frac{7}{4}=\pm\frac{\sqrt{17}}{4}$ | **7** Square root both sides. Remember that the square root of a value gives two answers. |
| $x=\pm\frac{\sqrt{17}}{4}+\frac{7}{4}$ | **8** Add $\frac{7}{4}$ to both sides. |
| So $x=\frac{7}{4}-\frac{\sqrt{17}}{4}$ or $x=\frac{7}{4}+\frac{\sqrt{17}}{4}$ | **9** Write down both the solutions. |

## Practice

**3** Solve by completing the square.

**a** $x^2 - 4x - 3 = 0$

**b** $x^2 - 10x + 4 = 0$

**c** $x^2 + 8x - 5 = 0$

**d** $x^2 - 2x - 6 = 0$

**e** $2x^2 + 8x - 5 = 0$

**f** $5x^2 + 3x - 4 = 0$

**4** Solve by completing the square.

**a** $(x - 4)(x + 2) = 5$

**b** $2x^2 + 6x - 7 = 0$

**c** $x^2 - 5x + 3 = 0$

**Hint**

Get all terms onto one side of the equation.

# Solving quadratic equations by using the formula

**A LEVEL LINKS**
**Scheme of work:** 1b. Quadratic functions – factorising, solving, graphs and the discriminants

## Key points

- Any quadratic equation of the form $ax^2 + bx + c = 0$ can be solved using the formula
  $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- If $b^2 - 4ac$ is negative then the quadratic equation does not have any real solutions.
- It is useful to write down the formula before substituting the values for $a$, $b$ and $c$.

## Examples

**Example 7** Solve $x^2 + 6x + 4 = 0$. Give your solutions in surd form.

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|---|---|
| $a = 1, b = 6, c = 4$<br>$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ | **1** Identify $a$, $b$ and $c$ and write down the formula.<br>Remember that $-b \pm \sqrt{b^2 - 4ac}$ is all over $2a$, not just part of it. |
| $x = \dfrac{-6 \pm \sqrt{6^2 - 4(1)(4)}}{2(1)}$ | **2** Substitute $a = 1$, $b = 6$, $c = 4$ into the formula. |
| $x = \dfrac{-6 \pm \sqrt{20}}{2}$ | **3** Simplify. The denominator is 2, but this is only because $a = 1$. The denominator will not always be 2. |
| $x = \dfrac{-6 \pm 2\sqrt{5}}{2}$ | **4** Simplify $\sqrt{20}$.<br>$\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$ |
| $x = -3 \pm \sqrt{5}$ | **5** Simplify by dividing numerator and denominator by 2. |
| So $x = -3 - \sqrt{5}$ or $x = \sqrt{5} - 3$ | **6** Write down both the solutions. |

**Example 8** Solve $3x^2 - 7x - 2 = 0$. Give your solutions in surd form.

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|---|---|
| $a = 3, b = -7, c = -2$<br>$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$<br><br>$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-2)}}{2(3)}$<br><br>$x = \frac{7 \pm \sqrt{73}}{6}$<br>So $x = \frac{7 - \sqrt{73}}{6}$ or $x = \frac{7 + \sqrt{73}}{6}$ | **1** Identify $a$, $b$ and $c$, making sure you get the signs right and write down the formula.<br>Remember that $-b \pm \sqrt{b^2 - 4ac}$ is all over $2a$, not just part of it.<br>**2** Substitute $a = 3$, $b = -7$, $c = -2$ into the formula.<br>**3** Simplify. The denominator is 6 when $a = 3$. A common mistake is to always write a denominator of 2.<br>**4** Write down both the solutions. |

# Practice

**5** Solve, giving your solutions in surd form.

**a** $3x^2 + 6x + 2 = 0$ **b** $2x^2 - 4x - 7 = 0$

**6** Solve the equation $x^2 - 7x + 2 = 0$

Give your solutions in the form $\frac{a \pm \sqrt{b}}{c}$, where $a$, $b$ and $c$ are integers.

**7** Solve $10x^2 + 3x + 3 = 5$

Give your solution in surd form.

**Hint**

Get all terms onto one side of the equation.

# Extend

**8** Choose an appropriate method to solve each quadratic equation, giving your answer in surd form when necessary.

**a** $4x(x - 1) = 3x - 2$

**b** $10 = (x + 1)^2$

**c** $x(3x - 1) = 10$

# Answers

**1** **a** $x = 0$ or $x = -\frac{2}{3}$ **b** $x = 0$ or $x = \frac{3}{4}$

**c** $x = -5$ or $x = -2$ **d** $x = 2$ or $x = 3$

**e** $x = -1$ or $x = 4$ **f** $x = -5$ or $x = 2$

**g** $x = 4$ or $x = 6$ **h** $x = -6$ or $x = 6$

**i** $x = -7$ or $x = 4$ **j** $x = 3$

**k** $x = -\frac{1}{2}$ or $x = 4$ **l** $x = -\frac{2}{3}$ or $x = 5$

**2** **a** $x = -2$ or $x = 5$ **b** $x = -1$ or $x = 3$

**c** $x = -8$ or $x = 3$ **d** $x = -6$ or $x = 7$

**e** $x = -5$ or $x = 5$ **f** $x = -4$ or $x = 7$

**g** $x = -3$ or $x = 2\frac{1}{2}$ **h** $x = -\frac{1}{3}$ or $x = 2$

**3** **a** $x = 2 + \sqrt{7}$ or $x = 2 - \sqrt{7}$ **b** $x = 5 + \sqrt{21}$ or $x = 5 - \sqrt{21}$

**c** $x = -4 + \sqrt{21}$ or $x = -4 - \sqrt{21}$ **d** $x = 1 + \sqrt{7}$ or $x = 1 - \sqrt{7}$

**e** $x = -2 + \sqrt{6.5}$ or $x = -2 - \sqrt{6.5}$ **f** $x = \frac{-3+\sqrt{89}}{10}$ or $x = \frac{-3-\sqrt{89}}{10}$

**4** **a** $x = 1 + \sqrt{14}$ or $x = 1 - \sqrt{14}$ **b** $x = \frac{-3+\sqrt{23}}{2}$ or $x = \frac{-3-\sqrt{23}}{2}$

**c** $x = \frac{5+\sqrt{13}}{2}$ or $x = \frac{5-\sqrt{13}}{2}$

**5** **a** $x = -1 + \frac{\sqrt{3}}{3}$ or $x = -1 - \frac{\sqrt{3}}{3}$ **b** $x = 1 + \frac{3\sqrt{2}}{2}$ or $x = 1 - \frac{3\sqrt{2}}{2}$

**6** $x = \frac{7+\sqrt{41}}{2}$ or $x = \frac{7-\sqrt{41}}{2}$

**7** $x = \frac{-3+\sqrt{89}}{20}$ or $x = \frac{-3-\sqrt{89}}{20}$

**8** **a** $x = \frac{7+\sqrt{17}}{8}$ or $x = \frac{7-\sqrt{17}}{8}$

**b** $x = -1 + \sqrt{10}$ or $x = -1 - \sqrt{10}$

**c** $x = -1\frac{2}{3}$ or $x = 2$