P1 Chapter 8 :: Binomial Expansion

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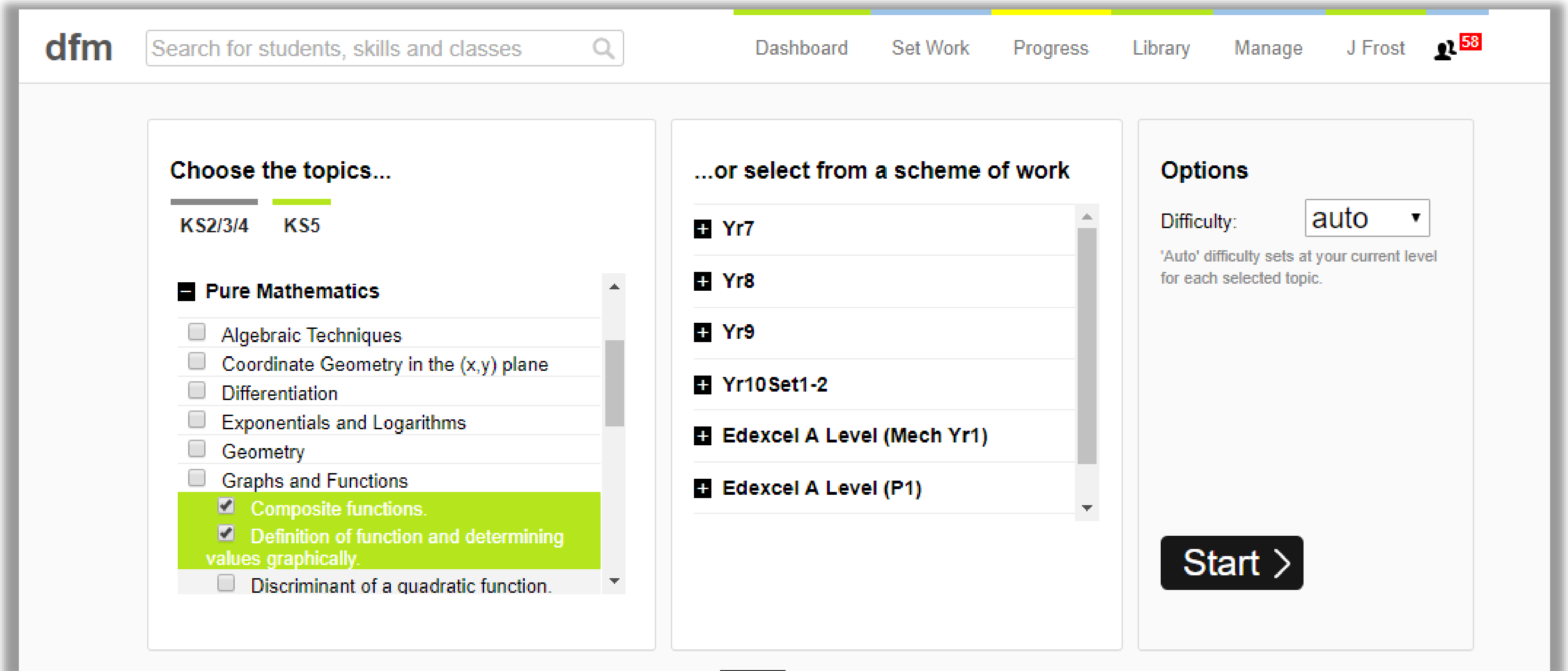

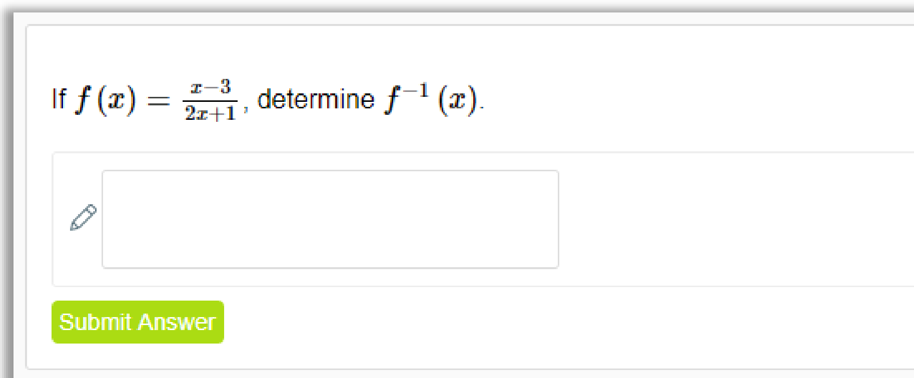

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Practise questions by chapter, including past paper Edexcel questions and extension questions (e.g. MAT).

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Chapter Overview

1:: Pascal's Triangle

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

2:: Factorial Notation

Given that $\binom{8}{3} = \frac{8!}{3!a!}$, find the value of a.

3:: Binomial Expansion

Find the first 4 terms in the binomial expansion of $(4+5x)^{10}$, giving terms in ascending powers of x.

4:: Using expansions for estimation

Use your expansion to estimate the value of 1.05^{10} to 5 decimal places.

Starter

a) Expand $(a+b)^0$
b) Expand $(a+b)^1$
c) Expand $(a+b)^2$
d) Expand $(a+b)^3$
e) Expand $(a+b)^4$

?
?
?
?
?

What do you notice about:

The coefficients:
The powers of a and b:

?
?

More on Pascal's Triangle

The second number of each row tells us what row we should use for an expansion.

So if we were expanding $(2 + x)^4$, the power is 4, so we use this row.

In Pascal's Triangle, each term (except for the 1s) is the sum of the two terms above.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Tip: I highly recommend memorising each row up to what you see here.

These are the coefficeents on each row.

We'll see later WHY each row gives us the coefficients in an expansion of $(a + b)^n$

More on Pascal's Triangle

$(a+b)^0$	1
$(a+b)^1$	1a 1b
$(a+b)^2$	$1a^2$ $2ab$ $1b^2$
$(a+b)^3$	$1a^3$ $3a^2b$ $3ab^2$ $1b^3$
$(a+b)^4$	$1a^4$ $4a^3b$ $6a^2b^2$ $4ab^3$ $1b^4$

Tip: The sum of powers all add to the original power.

← Powers of b decrease

Powers of a decrease →

Example

Find the expansion of $(2+3x)^4$

Next have descending or ascending powers of one of the terms, going between 0 and 4 (note that if the power is 0, the term is 1, so we need not write it).

$$
\begin{aligned}
(2+3x)^4 = \; & 1\,(2^4) \\
& +4\,(2^3)(3x)^1 \\
& +6\,(2^2)(3x)^2 \\
& +4\,(2^1)(3x)^3 \\
& +1\qquad\;\;(3x)^4
\end{aligned}
$$

First fill in the correct row of Pascal's triangle.

And do the same with the second term but with powers going the opposite way, noting again that the 'power of 0' term does not appear.

Simplify each term (ensuring any number in a bracket is raised to the appropriate power)

$$= 16 + 96x + 216x^2 + 216x^3 + 81x^4$$

Tip: Initially write *one line per term* for your expansion (before you simplify at the end), as we have done above. There will be less faffing trying to ensure you have enough space for each term.

Another Example

$(1-2x)$ is the same as $(1+(-2x))$, so we expand as before, but use $-2x$ for the second term.

$$
\begin{aligned}
(1-2x)^3 = \quad & 1\ (1^3) \\
& +3\ (1^2)(-2x)^1 \\
& +3\ (1)\ \ (-2x)^2 \\
& +1 \qquad\ \ (-2x)^3
\end{aligned}
$$

$$= 1 - 6x + 12x^2 - 8x^3$$

Tip: If one of the terms in the original bracket is negative, the terms in your expansion will <u>oscillate between positive and negative</u>. If they don't (e.g. two consecutive negatives), you're done something wrong!

Getting a single term in the expansion

The coefficient of x^2 in the expansion of $(2 - cx)^5$ is 720.
Find the possible value(s) of the constant c.

The '5' row in Pascal's triangle is 1 5 10 10 5 1. If we count the 1 as the '0th term', we want the 2nd term, which is 10.

Since we want the x^2 term:

- The power of $(-cx)$ must be ?
- The power of 2 must be ?

Therefore term is: ?

Test Your Understanding

Edexcel C2

(*a*) Find the first 3 terms, in ascending powers of x, of the binomial expansion of

$$(2 + kx)^7$$

where k is a constant. Give each term in its simplest form. **(4)**

Given that the coefficient of x^2 is 6 times the coefficient of x,

(*b*) find the value of k. **(2)**

(a)	?
(b)	?

Exercise 8A

Pearson Pure Mathematics Year 1/AS
Pages 160-161

Extension

1 *[MAT 2009 1J]*
The number of pairs of positive integers x, y which solve the equation:

$$x^3 + 6x^2y + 12xy^2 + 8y^3 = 2^{30}$$

is:

A) 0
B) 2^6
C) $2^9 - 1$
D) $2^{10} + 2$

?

Factorial and Choose Function

Notation:

- $Power\ of\ the\ expansion = n$

- $Position\ in\ the\ row\ of\ coefficients = r$

- $Row\ 5$: 1 5 10 10 5 1
- r = 0 r = 1 r = 2 r = 3 r = 4 r = 5

Factorial and Choose Function

✎ $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$

said "n factorial", is the number of ways of arranging n objects in a line.

For example, suppose you had three letters, A, B and C, and wanted to arrange them in a line to form a 'word', e.g. ACB or BAC.

- There are 3 choices for the first letter.
- There are then 2 choices left for the second letter.
- There is then only 1 choice left for the last letter.

There are therefore $3 \times 2 \times 1 = 3! = 6$ possible combinations.
Your calculator can calculate a factorial using the $x!$ button.

✎ ${}^nCr = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

said "n choose r", is the number of ways of 'choosing' r things from n, such that the order in our selection does not matter.
These are also known as **binomial coefficients**.

For example, if you are a football team captain and need to choose 4 people from amongst 10 in your class, there are $\binom{10}{4} = \frac{10!}{4!6!} = 210$ possible selections.

(Note: the $\binom{10}{4}$ notation is preferable to ${}^{10}C_4$)

Use the nCr button on your calculator (your calculator input should display "10C4")

Examples

Calculate the value of the following. You may use the factorial button, but not the nCr button.

a) $5!$

b) $\binom{5}{3}$

c) $0!$

d) $\binom{20}{1}$

e) $\binom{20}{0}$

f) $\binom{20}{2}$

g) $\binom{20}{18}$

a ?

b ?

c ?

d ?

e ?

f ?

g ?

Why do we care?

If the power in the binomial expansion is large, e.g. $(x+3)^{20}$, it is no longer practical to go this far down Pascal's triangle. We can instead use the choose function to get numbers from anywhere within the triangle. We'll practise doing this after the next exercise.

1

Notice: The top number matches the row number. The bottom number goes from 0 and eventually matches the top number. It's easy to see from the symmetry of Pascal's Triangle that $\binom{4}{1} = \binom{4}{3}$ for example.

1 3 3 1

1 4 6 4 1

0^{th} row

1^{st} row

2^{nd} row

3^{rd} row

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Textbook Note: The textbook refers to the top row as the "1^{st} row" and the first number in each row as the "1^{st} entry". This might sound sensible, but is against accepted practice: It makes much more sense that the row number matches the number at the top of the binomial coefficient, and the entry number matches the bottom number. We therefore call the top row the "0^{th} row" and the first entry of each row the "0^{th} entry".

So the kth entry of the nth row of Pascal's Triangle is therefore a nice clean $\binom{n}{k}$, not $\binom{n-1}{k-1}$ as suggested by the textbook.

Extra Cool Stuff

(You are not required to know this, but it is helpful for STEP)

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

We earlier saw that each entry of Pascal's Triangle is the sum of the two above it. Thus for example:

$$\binom{3}{1} + \binom{3}{2} = \binom{4}{2}$$

More generally:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

This is known as Pascal's Rule.

Informal proof of Pascal's Rule:

Suppose I have n items and I have to choose k of them. Clearly there's $\binom{n}{k}$ possible selections.

But we could also find the number of selections by considering the first item of the n available:

- It might be chosen. If so, we have $k-1$ items left to choose from amongst the $n-1$ remaining. That's $\binom{n-1}{k-1}$ possible selections.
- Otherwise it is not chosen. We still have k items to choose, from amongst the remaining $n-1$ items. That's $\binom{n-1}{k}$ possible selections.

Thus in total there are $\binom{n-1}{k-1} + \binom{n-1}{k}$ possible selections.

Exercise 8B

Pearson Pure Mathematics Year 1/AS

Pages 162

Using Binomial Coefficients to Expand

In the previous section we learnt about the ‘choose’ function and how this related to Pascal’s Triangle.

1 5 10 10 5 1

$$\binom{5}{0}\binom{5}{1}\binom{5}{2}\binom{5}{3}\binom{5}{4}\binom{5}{5}$$

Why do rows of Pascal’s Triangle give us the coefficients in a Binomial Expansion?

One possible selection of terms from each bracket.

Consider: $(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$

Each term of the expansion involves picking one term from each bracket.

How many times will a^3b^2 appear in the expansion?

?

Using Binomial Coefficients to Expand

$\mathbb{N}$ is the set of natural numbers, i.e. positive integers. This formula is only valid for positive integers n. In Year 2 you will see how to deal with fractional/negative n.

✎ The binomial expansion, when $n \in \mathbb{N}$:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{r} a^{n-r} b^r + \cdots + b^n$$

Find the first 4 terms in the expansion of $(3x+1)^{10}$, in ascending powers of x.

?

Test Your Understanding

Find the first 3 terms in the expansion of $\left(2 - \frac{1}{3}x\right)^7$, in ascending powers of x.

?

Exercise 8C

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Extension

1 [AEA 2013 Q1a] In the binomial expansion of $\left(1+\frac{12n}{5}x\right)^n$ the coefficients of x^2 and x^3 are equal and non-zero. Find the possible values of n.

?

Hint: Remember that
$$\binom{n}{2}=\frac{n(n-1)}{2!}$$
Can you similarly simplify $\binom{n}{3}$ using $\binom{n}{r}=\frac{n!}{r!(n-r)!}$?

2 [STEP I 2010 Q5a] By considering the expansion of $(1+x)^n$, where n is a positive integer, or otherwise, show that:

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$$

?

Reflection: This means that the sum of each row in Pascal's Triangle gives successive powers of 2.

Getting a single term in the expansion

In the expansion of $(a+b)^n$ the general term is given by $\binom{n}{r} a^{n-r} b^r$

Expression	Power of x in term wanted.	Term in expansion
$(a+x)^{10}$	3	?
$(2x-1)^{75}$	50	?
$(3-x)^{12}$	7	?
$(3x+4)^{16}$	3	?

Getting a single term in the expansion

The coefficient of x^4 in the expansion of $(1+qx)^{10}$ is 3360.
Find the possible value(s) of the constant q.

Term is:

?

Therefore:

?

Test Your Understanding

In the expansion of $(1 + ax)^{10}$, where a is a non-zero constant the coefficient of x^3 is double the coefficient of x^2. Find the value of a.

?

Exercise 8D

Pearson Pure Mathematics Year 1/AS
Pages 166-167

Extension

1 *[MAT 2014 1G]* Let n be a positive integer. The coefficient of x^3y^5 in the expansion of $(1 + xy + y^2)^n$ equals:

A) n
B) 2^n
C) $\binom{n}{3}\binom{n}{5}$
D) $4\binom{n}{4}$
E) $\binom{n}{8}$

?

2 [STEP I 2013 Q6] By considering the coefficient of x^r in the series for $(1 + x)(1 + x)^n$, or otherwise, obtain the following relation between binomial coefficients:

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$$

?

Estimating Powers

Edexcel C2 Jan 2012 Q3

(*a*) Find the first 4 terms of the binomial expansion, in ascending powers of x, of

$$\left(1+\frac{x}{4}\right)^8,$$

giving each term in its simplest form.

(4)

(*b*) Use your expansion to estimate the value of $(1.025)^8$, giving your answer to 4 decimal places.

(3)

Tip: Use your calculator to compare against the exact value of 1.025^8.

a

?

b

?

Test Your Understanding

Edexcel C2 Jan 2008 Q3

(*a*) Find the first 4 terms of the expansion of $\left(1+\frac{x}{2}\right)^{10}$ in ascending powers of x, giving each term in its simplest form.

(4)

(*b*) Use your expansion to estimate the value of $(1.005)^{10}$, giving your answer to 5 decimal places.

(3)

(a)

?

(b)

?

Exercise 8E

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