P2 Chapter 3 :: Sequences & Series

jfrost@tiffin.kingston.sch.uk

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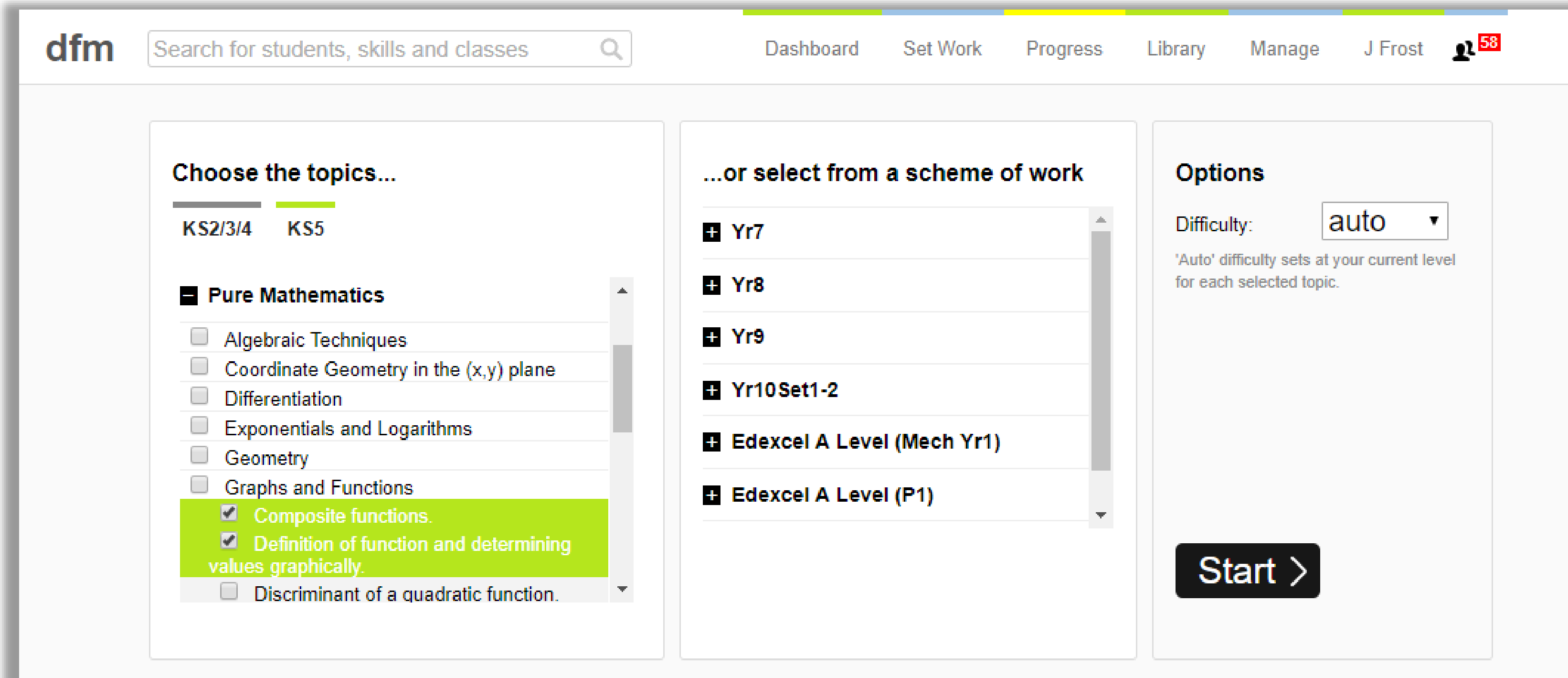

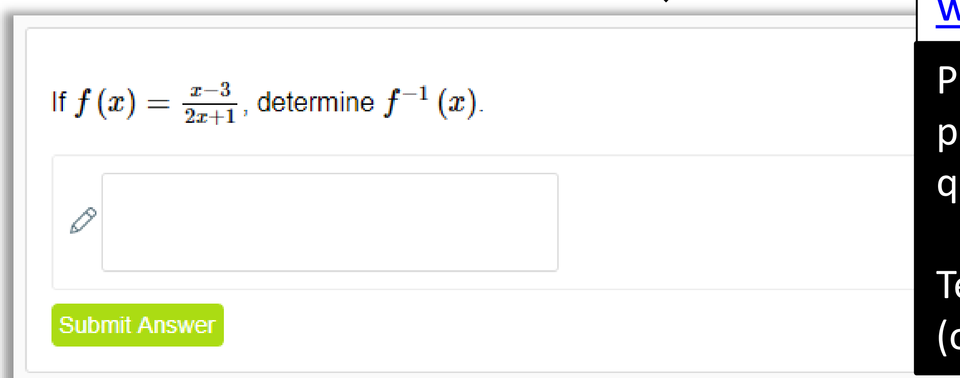

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Chapter Overview

1:: Sequences

2:: Arithmetic Series

Determine the value of $2 + 4 + 6 + \cdots + 100$

3:: Geometric Series

The first term of a geometric sequence is 3 and the second term 1. Find the sum to infinity.

4:: Sigma Notation

Determine the value of

$$\sum_{r=1}^{100} (3r + 1)$$

5:: Recurrence Relations

If $a_1 = k$ and $a_{n+1} = 2a_n - 1$, determine a_3 in terms of k.

NEW TO A LEVEL 2017!

Identifying whether a sequence is increasing, decreasing, or periodic.

Sequences

A sequence is an ordered set of mathematical objects. Each element in the sequence is called a term.

?

$$\overset{+3}{\frown}\ \overset{+3}{\frown}\ \overset{+3}{\frown}$$

$$2, 5, 8, 11, 14, \ldots$$

This is a:

?

An arithmetic sequence is one which has a common difference between terms.

?

$$\overset{\times 2}{\frown}\ \overset{\times 2}{\frown}\ \overset{\times 2}{\frown}$$

$$3, 6, 12, 24, 48, \ldots$$

?

Sequences

1, 1, 2, 3, 5, 8, …

?

The fundamentals of sequences

u_n ? Meaning

n ? Meaning

$n = \boxed{?}$

$u_{\boxed{?}} = \boxed{?}$

$$2, 5, 8, 11, 14, \ldots$$

n^{th} term of an arithmetic sequence

An arithmetic sequence is one which has a common difference between terms.

We use a to denote the **first term**. d is the **difference** between terms, and n is the **position** of the term we're interested in. Therefore:

1st Term	2nd Term	3rd Term	...	n^{th} term
?	?	?	...	?

n^{th} term of arithmetic sequence:

$$u_n = a + (n-1)d$$

n^{th} term of an arithmetic sequence

n^{th} term of arithmetic sequence:

$$u_n = a + (n-1)d$$

Example 1

The nth term of an arithmetic sequence is $u_n = 55 - 2n$.

a) Write down the first 3 terms of the sequence.

b) Find the first term in the sequence that is negative.

a) ?

b) ?

Example 2

Find the nth term of each arithmetic sequence.

a) 6, 20, 34, 48, 62

b) 101, 94, 87, 80, 73

Tip: Always write out $a =, d =, n =$ first.

a) ?

b) ?

Further Examples

[Textbook] A sequence is generated by the formula $u_n = an + b$ where a and b are constants to be found.
Given that $u_3 = 5$ and $u_8 = 20$, find the values of the constants a and b.

?

For which values of x would the expression $-8, x^2$ and $17x$ form the first three terms of an arithmetic sequence.

?

Test Your Understanding

Edexcel C1 May 2014(R) Q10

Xin has been given a 14 day training schedule by her coach.

Xin will run for A minutes on day 1, where A is a constant.

She will then increase her running time by $(d + 1)$ minutes each day, where d is a constant.

(a) Show that on day 14, Xin will run for

$$(A + 13d + 13) \text{ minutes.}$$

(2)

Yi has also been given a 14 day training schedule by her coach.

Yi will run for $(A - 13)$ minutes on day 1.

She will then increase her running time by $(2d - 1)$ minutes each day.

Given that Yi and Xin will run for the same length of time on day 14,

(b) find the value of d.

(3)

(a). ?

(b) ?

Exercise 3A

Pearson Pure Mathematics Year 2/AS
Pages 61-62

Extension

1 [STEP I 2004 Q5] The positive integers can be split into five distinct arithmetic progressions, as shown:

A: 1, 6, 11, 16, …
B: 2, 7, 12, 17, …
C: 3, 8, 13, 18, …
D: 4, 9, 14, 19, …
E: 5, 10, 15, 20, …

Write down an expression for the value of the general term in each of the five progressions. Hence prove that the sum of any term in B and any term in C is a term in E.

Prove also that the square of every term in B is a term in D. State and prove a similar claim about the square of every term in C.

i) Prove that there are no positive integers x and y such that $x^2 + 5y = 243723$
ii) Prove also that there are no positive integers x and y such $x^4 + 2y^4 = 26081974$

?

Series

A **series** is a <u>**sum**</u> of terms in a sequence.
You will encounter 'series' in many places in A Level:

Arithmetic Series (this chapter!)

Sum of terms in an arithmetic sequence.

$$2 + 5 + 8 + 11$$

Binomial Series (Later in Year 2)

You did Binomial expansions in Year 1. But when the power is negative or fractional, we end up with an infinite series.

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{6}x^3 - \cdots$$

Taylor Series (Further Maths)

Expressing a function as an infinite series, consisting of polynomial terms.

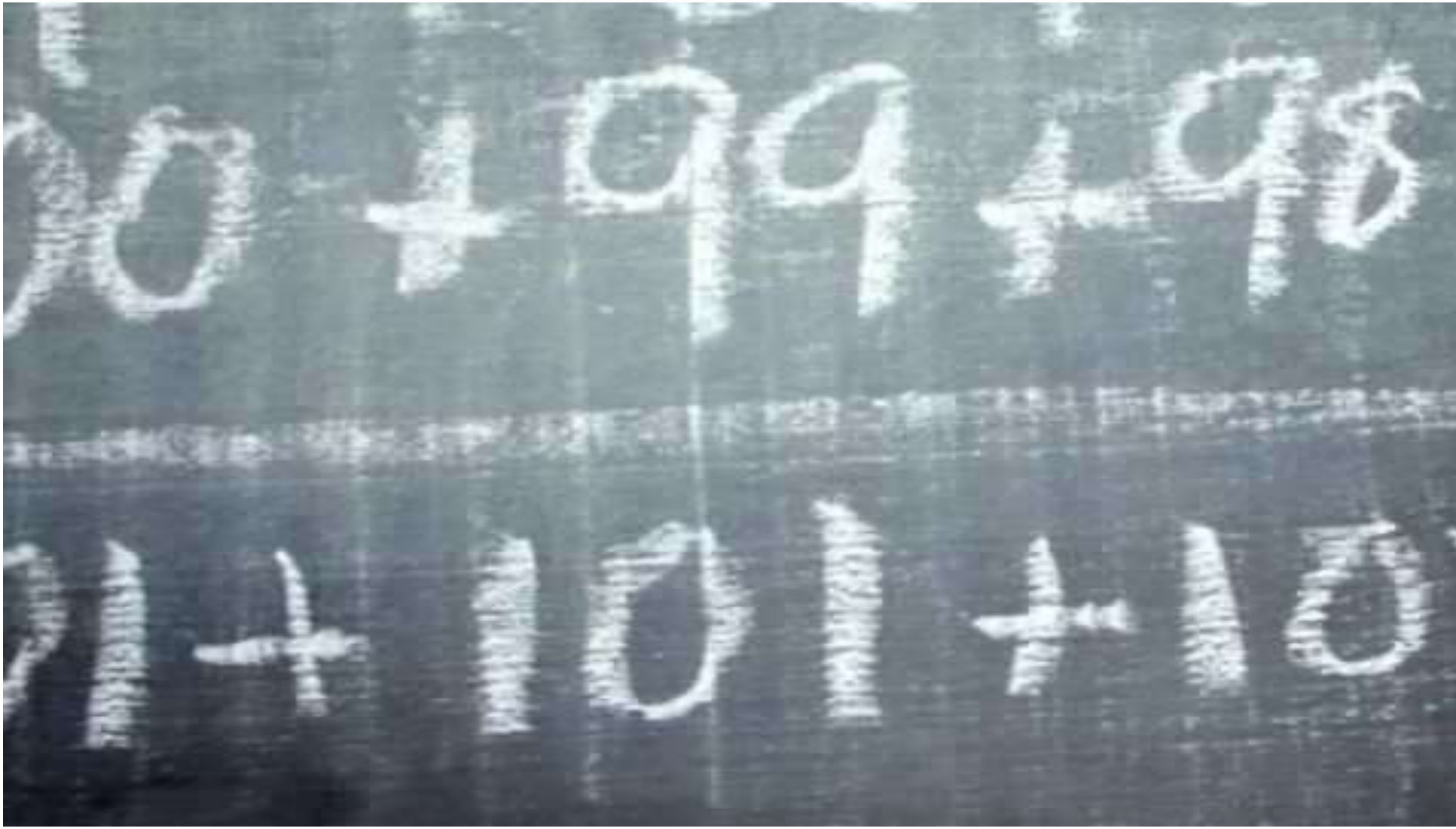
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Extra Notes: A 'series' usually refers to an <u>infinite</u> sum of terms in a sequence. If we were just summing some finite number of them, we call this a <u>partial sum of the series</u>.

e.g. The '*Harmonic Series*' is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which is infinitely many terms. But a we could get a partial sum, e.g. $H_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$

However, in this syllabus, the term 'series' is used to mean either a finite or infinite addition of terms.

Terminology: A '*power series*' is an infinite polynomial with increasing powers of x. There is also a chapter on power series in the Further Stats module.



https://www.youtube.com/watch?v=Prc7h8lyDXg

Arithmetic Series

n^{th} term

$$u_n = a + (n-1)d$$

Sum of first n terms

?

Let's prove it!

Example:

Take an arithmetic sequence $2, 5, 8, 11, 14, 17, \dots$

$$S_5 = 2 \quad + 5 \quad + 8 + 11 + 14$$

Reversing:

$$S_5 = 14 + 11 + 8 + 5 \quad + 2$$

The idea is that each pair of terms, at symmetrically opposite ends, adds to the same number.

Adding these:

$$2S_5 = 16 + 16 + 16 + 16 + 16 = 16 \times 5 = 80$$

$$\therefore S_5 = 40$$

Proving more generally:

$$S_n = \quad a \quad + (a+d) + (a+2d) + \cdots \quad + (a+(n-1)d)$$

$$S_n = (a+(n-1)d) + +(a+2d) + (a+d) + a$$

Adding:

$$2S_n = (2a+(n-1)d) + \cdots + (2a+(n-1)d) = n(2a+(n-1)d)$$

$$\therefore S_n = \frac{n}{2}(2a+(n-1)d)$$

Exam Note: The proof <u>has been an exam question before</u>. It's also a university interview favourite!

Alternative Formula

$$a + (a + d) + \cdots + L$$

Suppose last term was L.
We saw earlier that each opposite pair of terms (first and last, second and second last, etc.) added to the same total, in this case $a + L$.

There are $\frac{n}{2}$ pairs, therefore:

✏

$$S_n = \frac{n}{2}(a + L)$$

Examples

Find the sum of the first 30 terms of the following arithmetic sequences…

1 $2+5+8+11+14\ldots$ $S_{30} =$?

2 $100+98+96+\cdots$ $S_{30} =$?

3 $p+2p+3p+\cdots$ $S_{30} =$?

Tips: Again, explicitly write out "$a = \cdots, d = \cdots, n = \cdots$". You're less likely to make incorrect substitutions into the formula.

Make sure you write $S_n = \cdots$ so you make clear to yourself (and the examiner) that you're finding the sum of the first n terms, not the nth term.

Find the greatest number of terms for the sum of $4+9+14+\cdots$ to exceed 2000.

?

Test Your Understanding

Edexcel C1 Jan 2012 Q9

9. A company offers two salary schemes for a 10-year period, Year 1 to Year 10 inclusive.

Scheme 1: Salary in Year 1 is £P.
Salary increases by £($2T$) each year, forming an arithmetic sequence.

Scheme 2: Salary in Year 1 is £(P + 1800).
Salary increases by £T each year, forming an arithmetic sequence.

(*a*) Show that the total earned under Salary Scheme 1 for the 10-year period is

$$£(10P + 90T).$$

(2)

For the 10-year period, the total earned is the same for both salary schemes.

(*b*) Find the value of T.

?

(4)

For this value of T, the salary in Year 10 under Salary Scheme 2 is £29 850.

(*c*) Find the value of P.

?

(3)

Exercise 3B

Pearson Pure Mathematics Year 2/AS
Pages 64-66

Extension

1 [MAT 2007 1J]
The inequality
$$(n+1)+(n^4+2)+(n^9+3)+\cdots$$
$$+(n^{10000}+100)>k$$
Is true for all $n \geq 1$. It follows that

A) $k < 1300$
B) $k^2 < 101$
C) $k \geq 101^{10000}$
D) $k < 5150$

?

2 [AEA 2010 Q2]
The sum of the first p terms of an arithmetic series is q and the sum of the first q terms of the same arithmetic series is p, where p and q are positive integers and $p \neq q$.
Giving simplified answers in terms of p and q, find

a) The common difference of the terms in this series,
b) The first term of the series,
c) The sum of the first $(p+q)$ terms of the series.

Solution on next slide.

3 [MAT 2008 1I]
The function $S(n)$ is defined for positive integers n by
$$S(n) = \text{sum of digits of } n$$
For example, $S(723) = 7 + 2 + 3 = 12$.
The sum
$$S(1) + S(2) + S(3) + \cdots + S(99)$$
equals what?

?

Solution to Extension Q2

[AEA 2010 Q2]
The sum of the first p terms of an arithmetic series is q and the sum of the first q terms of the same arithmetic series is p, where p and q are positive integers and $p \neq q$.
Giving simplified answers in terms of p and q, find

a) The common different of the terms in this series,
b) The first term of the series,
c) The sum of the first $(p+q)$ terms of the series.

(a)	$q=\frac{p}{2}(2a+(p-1)d)$ and $p=\frac{q}{2}(2a+(q-1)d)$	M1 A1	Attempt one sum formula Both correct expressions
	$2\left(\frac{q}{p}-\frac{p}{q}\right)=d(p-1-q+1)$	dM1 A1	Eliminate a. Dep on 1st M1 Must use 2 indep. eqns Correct elimination of a
	$d=\frac{2(q^2-p^2)}{pq(p-q)}$; $\underline{d=\frac{-2(p+q)}{pq}}$	A1 (5)	Correct simplified $d=$
(b)	$2a=\frac{2q}{p}+\frac{(p-1)2(q+p)}{pq}$; $a=\frac{q^2(q-1)-p^2(p-1)}{pq(q-p)}$	M1	Substitute for d in a correct sum formula i.e. eqn in a only
		dM1	Rearrange to $a=$. Dep M1
	$\underline{\frac{q^2+qp+p^2-p-q}{pq}}$ or $\underline{\frac{q^2+(p-1)(q+p)}{pq}}$ or $\underline{\frac{p^2+(q-1)(q+p)}{pq}}$	A1 (3)	Correct single fraction with denom $=pq$
(c)	$S_{p+q}=\frac{p+q}{2}\left(\frac{2q}{p}+\frac{(p-1)2(q+p)}{pq}+\frac{-2(p+q)}{pq}(p+q-1)\right)$	M1	Attempt sum formula with $n=(p+q)$ and ft their a and d
	$=\frac{p+q}{2}\left[\frac{2(q^2+qp+p^2-p-q)}{pq}-\frac{2(p+q-1)(p+q)}{pq}\right]$	M1	Attempt to simplify-denominator $=pq$ or $2pq$
	$\frac{p+q}{pq}[-pq]=-[p+q]$	A1 (3) [11]	A1for $-(p+q)$ (S+ for concise simplification/factorising)

Recap of Arithmetic vs Geometric Sequences

?

$$2 \xrightarrow{+3} 5 \xrightarrow{+3} 8 \xrightarrow{+3} 11, 14, \ldots$$

This is a:

?

?

$$3 \xrightarrow{\times 2} 6 \xrightarrow{\times 2} 12 \xrightarrow{\times 2} 24, 48, \ldots$$

?

✎ A geometric sequence is one in which there is a **common ratio** between terms.

Quickfire Common Ratio

Identify the common ratio r:

1	$1, 2, 4, 8, 16, 32, \ldots$	$r =$?
2	$27, 18, 12, 8, \ldots$	$r =$?
3	$10, 5, 2.5, 1.25, \ldots$	$r =$?
4	$5, -5, 5, -5, 5, -5, \ldots$	$r =$?
5	$x, -2x^2, 4x^3$	$r =$?
6	$1, p, p^2, p^3, \ldots$	$r =$?
7	$4, -1, 0.25, -0.0625, \ldots$	$r =$?

An **alternating sequence** is one which oscillates between positive and negative.

n^{th} term

Arithmetic Sequence	Geometric Sequence
$u_n =$?	$u_n =$?

Determine the 10$^{\text{th}}$ and n^{th} terms of the following:

3, 6, 12, 24, … → ?

40, -20, 10, -5, … → ?

Tip: As before, write out $a =$ and $r =$ first before substituting.

Further Example

[Textbook] The second term of a geometric sequence is 4 and the 4th term is 8. The common ratio is positive. Find the exact values of:

a) The common ratio.
b) The first term.
c) The 10th term.

?

Tip: Explicitly writing $u_2 = 4$ first helps you avoid confusing the n^{th} term with the 'sum of the first n terms' (the latter of which we'll get onto).

Further Example

[Textbook] The numbers $3, x$ and $x + 6$ form the first three terms of a positive geometric sequence. Find:

a) The value of x.
b) The 10^{th} term in the sequence.

Hint: You're told it's a geometric sequence, which means the ratio between successive terms must be the same. Consequently $\frac{u_2}{u_1} = \frac{u_3}{u_2}$

?

Exam Note: This kind of question has appeared in the exam multiple times.

n^{th} term with inequalities

[Textbook] What is the first term in the geometric progression $3, 6, 12, 24, \ldots$ to exceed 1 million?

?

Test Your Understanding

All the terms in a geometric sequence are positive.
The third term of the sequence is 20 and the fifth term 80. What is the 20th term?

?

The second, third and fourth term of a geometric sequence are the following:

$$x, \qquad x+6, \qquad 5x-6$$

a) Determine the possible values of x.
b) Given the common ratio is positive, find the common ratio.
c) Hence determine the possible values for the first term of the sequence.

?

Exercise 3C

Pearson Pure Mathematics Year 2/AS
Pages 69-70

Sum of the first n terms of a geometric **series**

Arithmetic Series

$S_n =$?

Geometric Series

$S_n =$?

Proof:

?

Exam Note: This once came up in an exam. And again is a university interview favourite!

Examples

Geometric Series

$$S_n = \frac{a(1-r^n)}{1-r}$$

Find the sum of the first 10 terms.

$3, 6, 12, 24, 48, \ldots$

$a =$? $r =$? $n =$?

$S_n =$?

$4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

$a =$? $, r =$? $, n =$?

$S_n =$?

Harder Example

Find the least value of n such that the sum of $1 + 2 + 4 + 8 + \cdots$ to n terms would exceed 2 000 000.

?

?

Test Your Understanding

Edexcel C2 June 2011 Q6

The second and third terms of a geometric series are 192 and 144 respectively.

For this series, find

(*a*) the common ratio,

(2)

(*b*) the first term,

(2)

(*d*) the smallest value of n for which the sum of the first n terms of the series exceeds 1000.

(4)

(a)	?
(b)	?
(d)	?

Exercise 3D

Pearson Pure Mathematics Year 2/AS
Pages 72-73

Extension

1 [MAT 2010 1B]
The sum of the first $2n$ terms of

$$1, 1, 2, \frac{1}{2}, 4, \frac{1}{4}, 8, \frac{1}{8}, 16, \frac{1}{16}, \ldots$$

is

A) $2^n + 1 - 2^{1-n}$
B) $2^n + 2^{-n}$
C) $2^{2n} - 2^{3-2n}$
D) $\frac{2^n - 2^{-n}}{3}$

?

Divergent vs Convergent

What can you say about the sum of each series up to infinity?

$1 + 2 + 4 + 8 + 16 + \ldots$

?

$1 - 2 + 3 - 4 + 5 - 6 + \ldots$

?

$1 + 0.5 + 0.25 + 0.125 + \ldots$

?

Definitely NOT in the A Level syllabus, and just for fun...

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

?

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

?

Sum to Infinity

$1 + 0.5 + 0.25 + 0.125 + \ldots$

Why did this infinite sum converge (to 2)...

$1 + 2 + 4 + 8 + 16 + \ldots$

...but this diverge to infinity?

- The infinite series will converge provided that $-1 < r < 1$ (which can be written as $|r| < 1$), because the terms will get smaller.
- Provided that $|r| < 1$, what happens to r^n as $n \to \infty$?

?

A geometric series is convergent if $|r| < 1$.

- How therefore can we use the $S_n = \frac{a(1-r^n)}{1-r}$ formula to find the sum to infinity, i.e. S_∞?

For a convergent geometric series,

$$S_\infty = \frac{a}{1-r}$$

Quickfire Examples

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

$\boldsymbol{a} =$? $\quad \boldsymbol{r} =$? $\quad \boldsymbol{S_\infty} =$?

$$27, -9, 3, -1, \ldots$$

$\boldsymbol{a} =$? $\quad \boldsymbol{r} =$? $\quad \boldsymbol{S_\infty} =$?

$$p, p^2, p^3, p^4, \ldots$$

$$where\ -1 < p < 1$$

$\boldsymbol{a} =$? $\quad \boldsymbol{r} =$? $\quad \boldsymbol{S_\infty} =$?

$$p, 1, \frac{1}{p}, \frac{1}{p^2}, \ldots$$

$\boldsymbol{a} =$? $\quad \boldsymbol{r} =$? $\quad \boldsymbol{S_\infty} =$?

Further Examples

[Textbook] The fourth term of a geometric series is 1.08 and the seventh term is 0.23328.

a) Show that this series is convergent.
b) Find the sum to infinity of this series.

[Textbook] For a geometric series with first term a and common ratio r, $S_4 = 15$ and $S_\infty = 16$.

a) Find the possible values of r.
b) Given that all the terms in the series are positive, find the value of a.

Fro Warning: The power is n in the S_n formula but $n-1$ in the u_n formula.

? a

? b

? a

? b

Test Your Understanding

Edexcel C2 May 2011 Q6

6. The second and third terms of a geometric series are 192 and 144 respectively.

For this series, find

(*a*) the common ratio, $r =$?
(2)

(*b*) the first term, $a =$?
(2)

(*c*) the sum to infinity, $S_\infty =$?
(2)

(*d*) the smallest value of *n* for which the sum of the first *n* terms of the series exceeds 1000.
(4)

?

Exercise 3E

Pearson Pure Mathematics Year 2/AS
Pages 75-76

Extension

1 [MAT 2006 1H] How many solutions does the equation

$$2 = \sin x + \sin^2 x + \sin^3 x + \sin^4 x + \cdots$$

have in the range $0 \leq x < 2\pi$

?

☠ [Frost] Determine the value of x where:

$$x = \frac{1}{1} + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \cdots$$

(Hint: Use an approach similar to proof of geometric S_n formula)

?

2 [MAT 2003 1F] Two players take turns to throw a fair six-sided die until one of them scores a six. What is the probability that the first player to throw the die is the first to score a six?

?

Sigma Notation

What does each bit of this expression mean?

The Greek letter, capital sigma, means 'sum'.

$$\sum_{r=1}^{5}(2r+1)$$

The numbers top and bottom tells us what r varies between. It goes up by 1 each time.

We work out this expression for each value of r (between 1 and 5), and add them together.

$$\underset{r=1}{=3}\quad \underset{r=2}{+5}\quad \underset{r=3}{+7}\quad \underset{r=4}{+9}\quad \underset{r=5}{+11}\quad = 35$$

If the expression being summed (in this case $2r+1$) is **linear**, we get an **arithmetic series**. We can therefore apply our usual approach of establishing a, d and n before applying the S_n formula.

Determining the value

	First few terms?	Values of a, n, d or r?	Final result?
$\sum_{n=1}^{7} 3n$	?	?	?
$\sum_{k=5}^{15} (10 - 2k)$	?	?	?
$\sum_{k=1}^{12} 5 \times 3^{k-1}$	?	?	?
$\sum_{k=5}^{12} 5 \times 3^{k-1}$	?		

Testing Your Understanding

Solomon Paper A

Evaluate

$$\sum_{r=10}^{30} (7 + 2r).$$ **(4)**

Method 1: Direct

?

Method 2: Subtraction

?

On your calculator

"Use of Technology" Monkey says:
The Classwiz and Casio Silver calculator has a Σ button.

Try and use it to find:

$$\sum_{k=5}^{12} 2 \times 3^k$$

Exercise 3F

Pearson Pure Mathematics Year 2/AS
Pages 77-78

Recurrence Relations

$$u_n = 2n^2 + 3$$

This is an example of a position-to-term sequence, because each term is based on the position n.

$$u_{n+1} = 2u_n + 4$$
$$u_1 = 3$$

We need the first term because the recurrence relation alone is not enough to know what number the sequence starts at.

But a term might be defined based on previous terms.
If u_n refers to the current term, u_{n+1} refers to the next term.
So the example in words says "the next term is twice the previous term + 4"

This is known as a term-to-term sequence, or more formally as a **recurrence relation**, as the sequence 'recursively' refers to itself.

Example

Important Note: With recurrence relation questions, the **<u>the sequence will likely not be arithmetic nor geometric</u>**. So your previous u_n and S_n formulae do not apply.

Edexcel C1 May 2013 (R)

6. A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1,$$

$$x_{n+1} = (x_n)^2 - kx_n, \qquad n \geq 1,$$

where k is a constant.

(*a*) Find an expression for x_2 in terms of k.

?

(1)

(*b*) Show that $x_3 = 1 - 3k + 2k^2$.

?

(2)

Given also that $x_3 = 1$,

(*c*) calculate the value of k.

?

(3)

(*d*) Hence find the value of $\sum_{n=1}^{100} x_n$.

?

(3)

Tip: When a Σ question comes up for recurrence relations, it will most likely be some kind of repeating sequence. Just work out the sum of terms in each repeating bit, and how many times it repeats.

Test Your Understanding

Edexcel C1 Jan 2012

4. A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1,$$

$$x_{n+1} = a x_n + 5, \qquad n \geq 1,$$

where a is a constant.

(*a*) Write down an expression for x_2 in terms of a.

?

(1)

(*b*) Show that $x_3 = a^2 + 5a + 5$.

?

(2)

Given that $x_3 = 41$

(*c*) find the possible values of a.

?

(3)

Combined Sequences

Sequences (or series) can be generated from a combination of both an arithmetic and a geometric sequence.

Example

4. (i) Show that $\sum_{r=1}^{16}(3 + 5r + 2^r) = 131\,798$ **(4)**

(ii) A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_{n+1} = \frac{1}{u_n}, \quad u_1 = \frac{2}{3}$$

Find the exact value of $\sum_{r=1}^{100} u_r$ **(3)**

Exercise 3G

Pearson Pure Mathematics Year 2/AS
Pages 80-81

1 *[AEA 2011 Q3]* A sequence $\{u_n\}$ is given by

$$u_1 = k$$
$$u_{2n} = u_{2n-1} \times p, \qquad n \geq 1$$
$$u_{2n+1} = u_{2n} \times q \qquad n \geq 1$$

(a) Write down the first 6 terms in the sequence.

(b) Show that $\sum_{r=1}^{2n} u_r = \frac{k(1+p)(1-(pq)^n)}{1-pq}$

(c) $[x]$ means the integer part of x, for example $[2.73] = 2, [4] = 4$.

Find $\sum_{r=1}^{\infty} 6 \times \left(\frac{4}{3}\right)^{\left[\frac{r}{2}\right]} \times \left(\frac{3}{5}\right)^{\left[\frac{r-1}{2}\right]}$

?

2 [MAT 2014 1H] The function $F(n)$ is defined for all positive integers as follows: $F(1) = 0$ and for all $n \geq 2$,

$F(n) = F(n-1) + 2$ if 2 divides n but 3 does not divide ,
$F(n) = F(n-1) + 3$ if 3 divides n but 2 does not divide n,
$F(n) = F(n-1) + 4$ if 2 and 3 both divide n
$F(n) = F(n-1)$ if neither 2 nor 3 divides n.

Then the value of $F(6000)$ equals what?

?

3 [MAT 2016 1G] The sequence (x_n), where $n \geq 0$, is defined by $x_0 = 1$ and $x_n = \sum_{k=0}^{n-1}(x_k)$ for $n \geq 1$

Determine the value of the sum

$$\sum_{k=0}^{\infty} \frac{1}{x_k}$$

Solution on next slide.

Solution to Extension Question 3

[MAT 2016 1G] The sequence (x_n), where $n \geq 0$, is defined by $x_0 = 1$ and
$x_n = \sum_{k=0}^{n-1}(x_k) \quad$ for $n \geq 1$
Determine the value of the sum

$$\sum_{k=0}^{\infty} \frac{1}{x_k}$$

Solution:

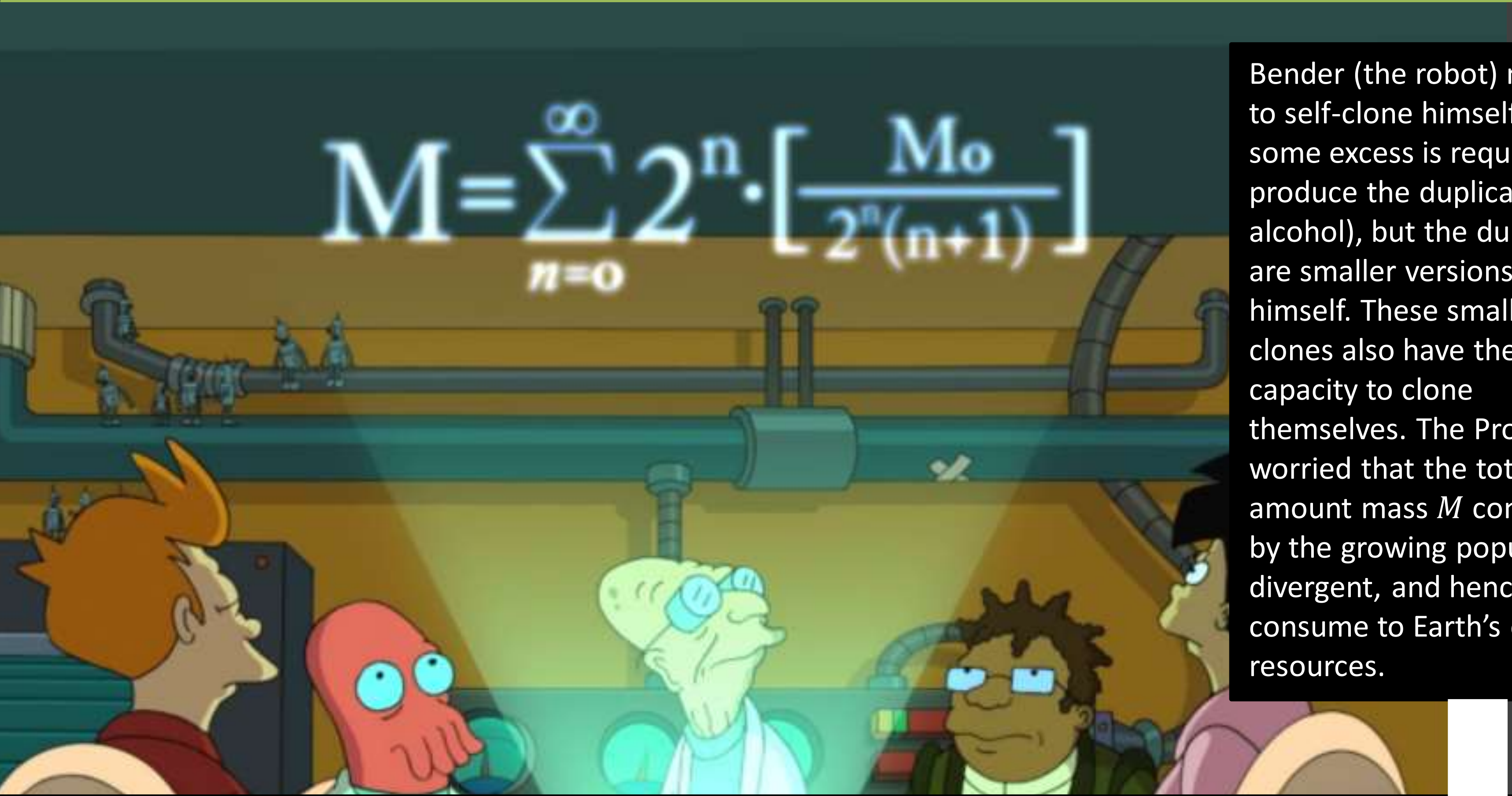
$$\boldsymbol{x_0 = 1}$$
$$\boldsymbol{x_1 = 1}$$
$$\boldsymbol{x_2 = 1 + 1 = 2}$$
$$\boldsymbol{x_3 = 1 + 1 + 2 = 4}$$

This is a geometric sequence from $\boldsymbol{x_1}$ onwards.

Therefore

$$\boldsymbol{\sum_{k=0}^{\infty} \frac{1}{x_k} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots}$$
$$\boldsymbol{= 1 + 2 = 3}$$

A somewhat esoteric Futurama joke explained



Bender (the robot) manages to self-clone himself, where some excess is required to produce the duplicates (e.g. alcohol), but the duplicates are smaller versions of himself. These smaller clones also have the capacity to clone themselves. The Professor is worried that the total amount mass M consumed by the growing population is divergent, and hence they'll consume to Earth's entire resources.

This simplifies to

$$M = M_0 \sum_{n=1}^{\infty} \frac{1}{n}$$

The sum $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$ is known as the **harmonic series**, which is divergent.

Increasing, decreasing and periodic sequences

A sequence is **strictly increasing** if the terms are always increasing, i.e. $u_{n+1} > u_n$ for all $n \in \mathbb{N}$.
e.g. $1, 2, 4, 8, 16, \ldots$

Textbook Error: It uses the term 'increasing' when it means 'strictly increasing'.

Similarly a sequence is **strictly decreasing** if $u_{n+1} < u_n$ for all $n \in \mathbb{N}$

A sequence is **periodic** if the terms repeat in a cycle. The **order** k of a sequence is **how often it repeats**, i.e. $u_{n+k} = u_n$ for all n.
e.g. 2, 3, 0, 2, 3, 0, 2, 3, 0, 2, … is periodic and has order 3.

[Textbook] For each sequence:
i) State whether the sequence is increasing, decreasing or periodic.
ii) If the sequence is periodic, write down its order.

a) $u_{n+1} = u_n + 3, \; u_1 = 7$?
b) $u_{n+1} = (u_n)^2, \quad u_1 = \frac{1}{2}$?
c) $u_{n+1} = \sin(90n°)$?

Exercise 3H

Pearson Pure Mathematics Year 2/AS

Pages 82-83

Modelling

Anything involving compound changes (e.g. bank interest) will form a geometric sequence, as there is a constant ratio between terms.
We can therefore use formulae such as S_n to solve problems.

[Textbook] Bruce starts a new company. In year 1 his profits will be £20 000. He predicts his profits to increase by £5000 each year, so that his profits in year 2 are modelled to be £25 000, in year 3, £30 000 and so on. He predicts this will continue until he reaches annual profits of £100 000. He then models his annual profits to remain at £100 000.

a) Calculate the profits for Bruce's business in the first 20 years.
b) State one reason why this may not be a suitable model.
c) Bruce's financial advisor says the yearly profits are likely to increase by 5% per annum. Using this model, calculate the profits for Bruce's business in the first 20 years.

a

? a

b

? b

c

? c

Geometric Modelling Example

[Textbook] A piece of A4 paper is folded in half repeatedly. The thickness of the A4 paper is 0.5 mm.

(a) Work out the thickness of the paper after four folds.

(b) Work out the thickness of the paper after 20 folds.

(c) State one reason why this might be an unrealistic model.

a

?

b

?

c

?

Test Your Understanding

Edexcel C2 Jan 2013 Q3

A company predicts a yearly profit of £120 000 in the year 2013. The company predicts that the yearly profit will rise each year by 5%. The predicted yearly profit forms a geometric sequence with common ratio 1.05.

(*a*) Show that the predicted profit in the year 2016 is £138 915.

(1)

(*b*) Find the first year in which the yearly predicted profit exceeds £200 000.

(5)

(*c*) Find the total predicted profit for the years 2013 to 2023 inclusive, giving your answer to the nearest pound.

(3)

(a) ? a

(b) ? b

A1

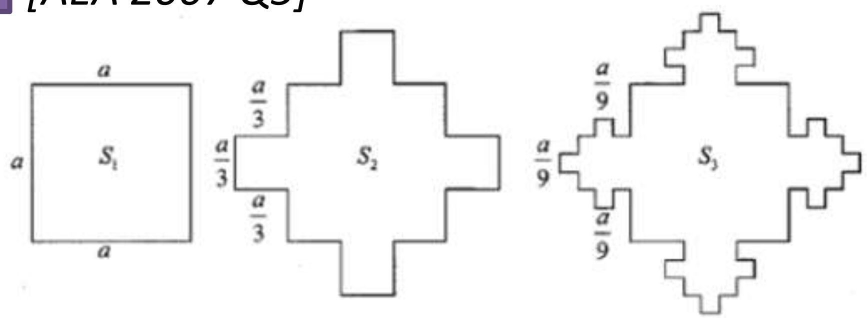
(c) ? c

Exercise 3I

Pearson Pure Mathematics Year 2/AS
Pages 84-86

Extension

1 *[AEA 2007 Q5]*



The figure shows part of a sequence $S_1, S_2, S_3, \ldots$, of model snowflakes. The first term S_1 consist of a single square of side a. To obtain S_2, the middle third of each edge is replaced with a new square, of side $\frac{a}{3}$, as shown. Subsequent terms are added by replacing the middle third of each external edge of a new square formed in the previous snowflake, by a square $\frac{1}{3}$ of the size, as illustrated by S_3.

a) Deduce that to form S_4, 36 new squares of side $\frac{a}{27}$ must be added to S_3.

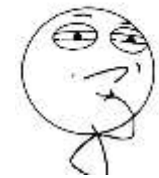
b) Show that the perimeters of S_2 and S_3 are $\frac{20a}{3}$ and $\frac{28a}{3}$ respectively.

c) Find the perimeter of S_n.

?

Just for your interest...

Tell me about this ‘**harmonic series**’...



A harmonic series is the infinite summation:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

We can also use H_k to denote the sum up to the k^{th} term (known as a *partial* sum), i.e.

$$H_k=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}$$

Is it convergent or divergent?
The terms we are adding gradually get smaller, so we might wonder if the series ‘**converges**’ to an upper limit (just like $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ approaches/converges to 2), or is infinitely large (i.e. ‘**diverges**’).

Proof

We can prove the series diverges by finding a value which is smaller, but that itself diverges:

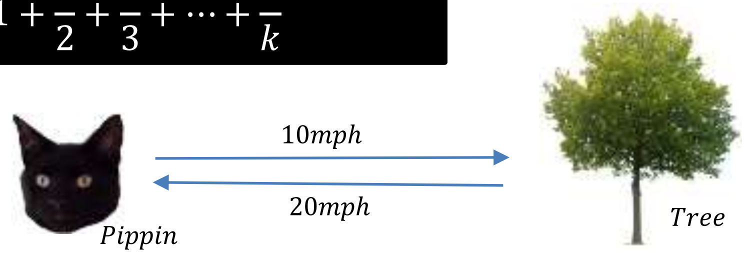
$$H=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots$$
$$x=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\cdots$$

Above we defined a new series x, such that we use one $\frac{1}{2}$, two $\frac{1}{4}s$, four $\frac{1}{8}s$, eight $\frac{1}{16}$s and so on. Each term is clearly less than (or equal to) each term in H. But it simplifies to:

$$x=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$$

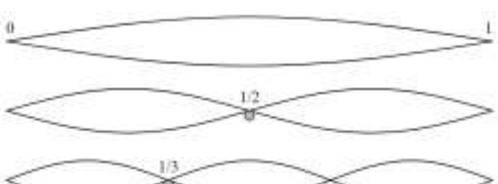
which diverges. Since $x<H$, H must also diverge.

It is also possible to prove that H_k is never an integer for any $k>1$.



Where does the name come from?

The ‘**harmonic mean**’ is used to find the **average of rates**. For example, if a cat runs to a tree at 10mph and back at 20mph, its average speed across the whole journey is $13\frac{1}{3}$ mph (not 15mph!).
Each term in the harmonic series is the harmonic mean of the two adjacent terms, e.g. $\frac{1}{3}$ is the harmonic mean of $\frac{1}{2}$ and $\frac{1}{4}$**.** The actual word ‘harmonic’ comes from music/physics and is to do with sound waves.



The **harmonic mean** of x and y is $\frac{2xy}{x+y}$.

Really cool applications:

The time Dr Frost solved a maths problem for John Lewis

My mum, who worked at John Lewis, was selling London Olympics 'trading cards', of which there were about 200 different cards to collect, and could be bought in packs. Her manager was curious how many cards you would have to buy on average before you collected them all. The problem was passed on to me…

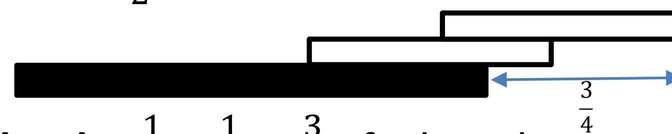
It is known as the '**coupon collector's problem**'.
The key is **dividing purchasing into separate stages where each stage involves getting the next unseen card**
There is an initial probability of 1 that a purchase results in an unseen card. There is then a $\frac{199}{200}$ chance the next purchase results in an unseen card. Reciprocating this probability gives us the number of cards on average, i.e. $\frac{200}{199}$, we'd have to buy until we get our second unseen card. Eventually, there is a $\frac{1}{200}$ chance of a purchase giving us the last unseen card, for which we'd have to purchase $\frac{200}{1} = 200$ cards on average. The total number of cards on average we need to buy is therefore:

$$\frac{200}{200} + \frac{200}{199} + \cdots + \frac{200}{2} + \frac{200}{1}$$
$$= 200\left(\frac{1}{200} + \frac{1}{199} + \cdots + \frac{1}{2} + 1\right)$$
$$= 200 \times H_{200} = 1176\ cards$$

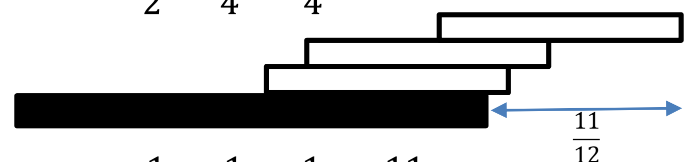
You're welcome John Lewis.

How far can a stack of n books protrude over the edge of a table?

1 book: $\frac{1}{2}$ a length.



2 books: $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ of a length.



3 books: $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{11}{12}$

Each of these additions is half the sum of the first n terms of the harmonic series.
e.g. 3 books: $\frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{12}$

The overhang for n books is therefore $\frac{1}{2}H_n$

Because the harmonic series is divergent, we can keep adding books to get an arbitrarily large amount of overhang. But note that the harmonic series increases **very slowly**. A stack of 10,000 books would only overhang by 5 book lengths!