P1 Chapter 13 :: Integration

jfrost@tiffin.kingston.sch.uk

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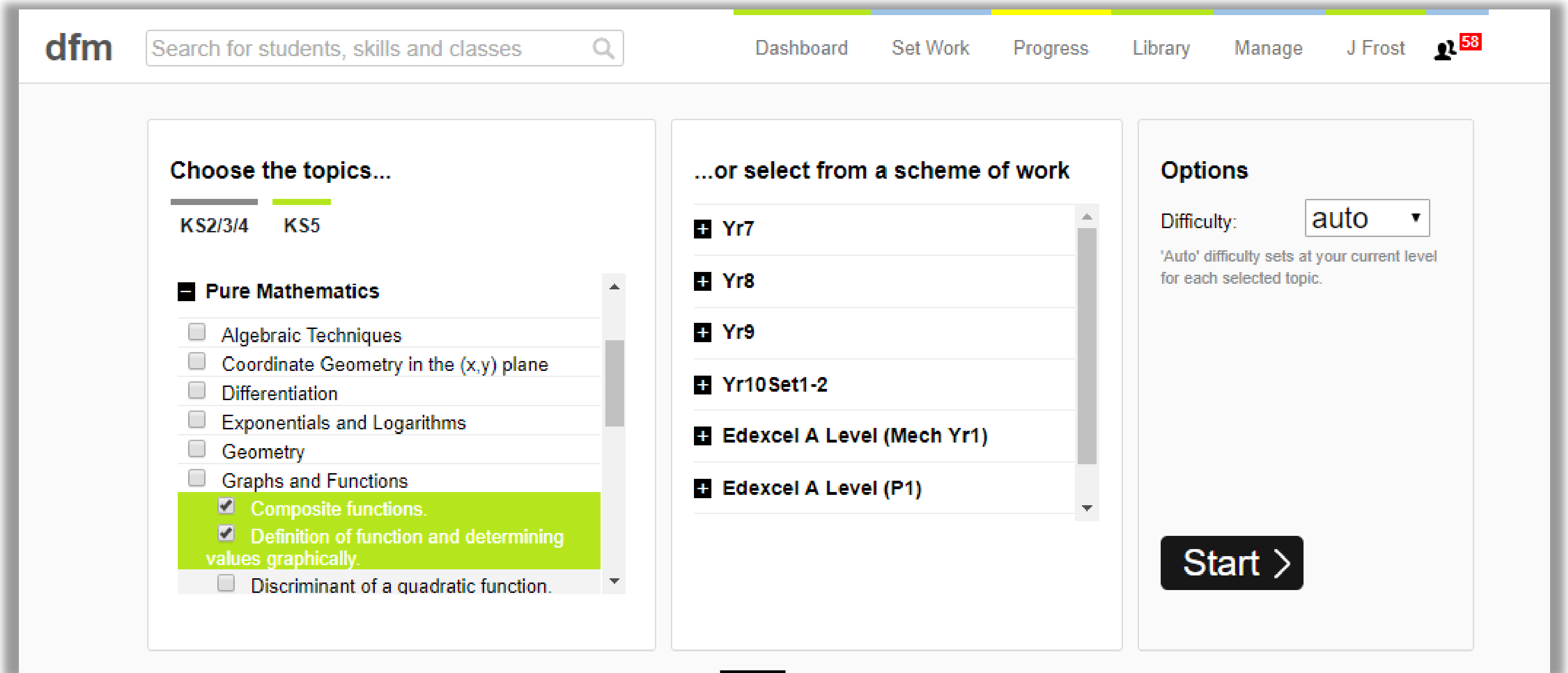

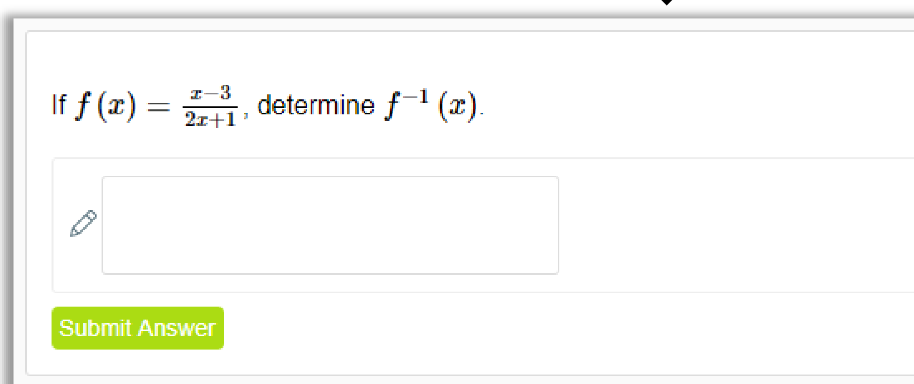

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Chapter Overview

This chapter is roughly divided into two parts: the first, **indefinite integration**, is the **opposite of differentiation**. The second, **definite integration**, allows us to **find areas under graphs** (as well as surface areas and volumes) or areas between two graphs.

1:: Find y given $\frac{dy}{dx}$

A curve has the gradient function

$$\frac{dy}{dx} = 3x + 1$$

If the curve goes through the point $(2,3)$, determine y.

2:: Evaluate definite integrals, and hence the area under a curve.

Find the area bounded between the curve with equation $y = x^2 - 2x$ and the x-axis.

3:: Find areas bound between two different lines.

Find the points of intersection of $y = x^2 - 4x + 3$ and $y = x + 9$, and hence find the area bound between the two lines.

Integrating x^n terms

Integration is the **opposite of differentiation**.
(For this reason it is also called '*antidifferentiation*')

Function

$$5x^3$$

multiply by power → reduce power by 1

Gradient Function

$$15x^2$$

? ← ?

However, there's one added complication…

Find y when $\frac{dy}{dx} = 3x^2$

?

Examples

Find y when:

$$\frac{dy}{dx} = 4x^3 \qquad y = ?$$

Exam Note: Historically 'C1' penalised the lack of $+c$, but modules thereafter didn't. You should **always** include it.

$$\frac{dy}{dx} = x^5 \qquad y = ?$$

You could also write as $\frac{x^6}{6}$. It's a matter of personal preference.

$$\frac{dy}{dx} = 3x^{\frac{1}{2}} \qquad y = ?$$

Tip: Many students are taught to write $\frac{3x^{\frac{3}{2}}}{\frac{3}{2}}$ (as does textbook!). This is ugly and students then often struggle to simplify it. Instead remember back to Year 7: **When you divide by a fraction, you multiply by the reciprocal.**

Textbook Error on Pg289: "When integrating polynomials", but Example 3 is not a polynomial because it has negative and fractional powers.

More Fractional Examples

Find y when:

$$\frac{dy}{dx} = \frac{4}{\sqrt{x}} = \boxed{?}$$

$$y = \boxed{?}$$

When we divide by $\frac{1}{2}$ we multiply by the reciprocal, i.e. 2.
Tip: I recommend doing in your head when the simplification would be simple.

$$\frac{dy}{dx} = 5x^{-2}$$

$$y = \boxed{?}$$

$$\frac{dy}{dx} = 4x^{\frac{2}{3}}$$

$$y = \boxed{?}$$

$$\frac{dy}{dx} = 10x^{-\frac{2}{7}}$$

$$y = \boxed{?}$$

Test Your Understanding

Find $f(x)$ when:

$f'(x) = 2x + 7$ $\quad$ $f(x) =$?

$f'(x) = x^2 - 1$ $\quad$ $f(x) =$?

$f'(x) = \dfrac{2}{x^7} =$? $\quad$ $f(x) =$?

$f'(x) = \sqrt[3]{x} =$? $\quad$ $f(x) =$?

$f'(x) = 33x^{\frac{5}{6}}$ $\quad$ $f(x) =$?

Note: In case you're wondering what happens if $\frac{dy}{dx} = \frac{1}{x} = x^{-1}$, the problem is that after adding 1 to the power, we'd be dividing by 0. You will learn how to integrate $\frac{1}{x}$ in Year 2.

Exercise 13A

Pearson Pure Mathematics Year 1/AS
Pages 289-290

Integration notation

The following notation could be used to differentiate an expression:

The dx here means differentiating "with respect to x".

$$\frac{d}{dx}(5x^2) = 10x$$

There is similarly notation for integrating an expression:

$$\int 10x \;\; dx = 5x^2 + c$$

"Integrate…"

"…this expression"

"…with respect to x"
(the dx is needed just as it was needed in the differentiation notation at the top of this slide)

This is known as **indefinite integration**, in contrast to definite integration, which we'll see later in the chapter.

It is called 'indefinite' because the exact expression is unknown (due to the $+c$).

Examples

Fro Note: The brackets are required if there's multiple terms.

Find $\int (x^{-\frac{3}{2}} + 2)\, dx$

?

Find $\int (6t^2 - 1)\, dt$

Note the dt instead of dx.

?

Find $\int (px^3 + q)\, dx$ where p and q are constants.

?

Textbook (Minor) Error: "*any other letters must be treated as constants*". Similar to the error in the differentiation chapter, it should read "*any other letters, which are either constants or variables independent of x, can be treated as numbers*". In $\int xy\, dx$, if y is a variable, we can only treat y as a constant if it is not dependent on x, i.e. there is not some equation relating y to x.

Test Your Understanding

Edexcel C1 May 2014(R) Q4b

Given that $y = 2x^5 + \frac{6}{\sqrt{x}}$, $x > 0$, find in their simplest form

(b) $\int y \, dx$ **(3)**

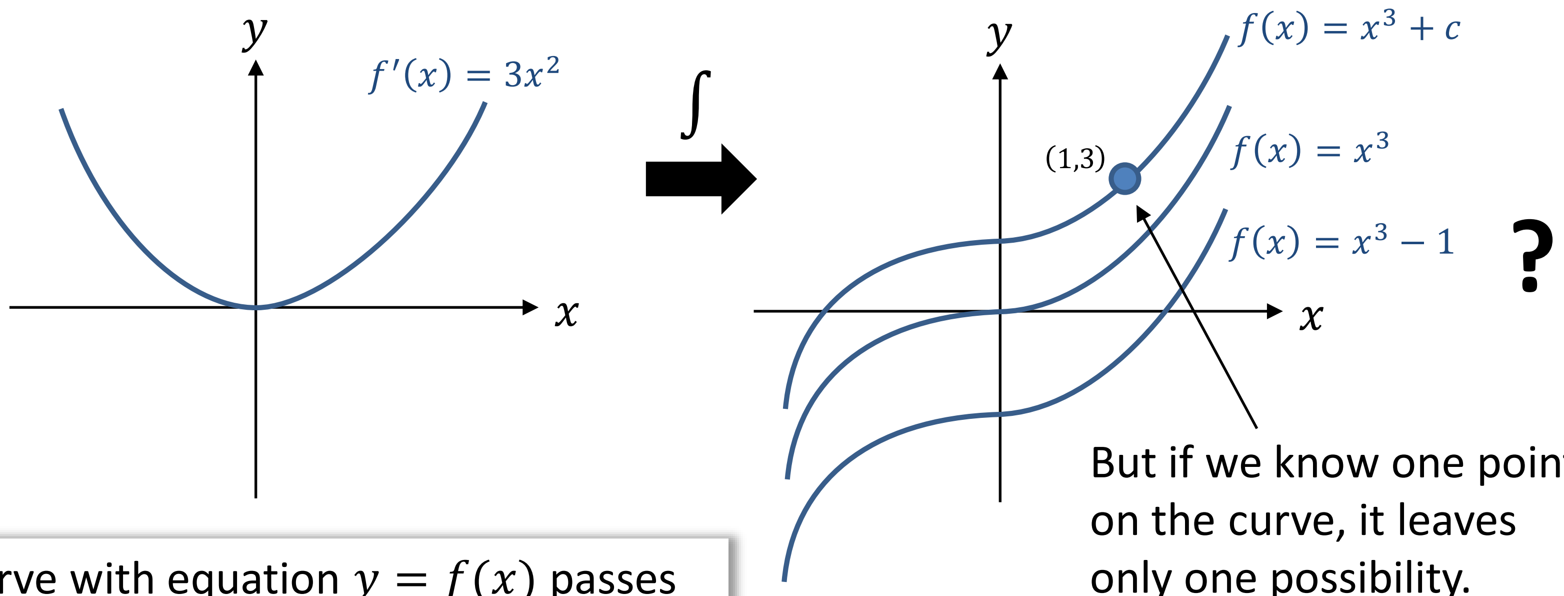
?

Exercise 13B

Pearson Pure Mathematics Year 1/AS
Pages 291-293

Finding constant of integration

Recall that when we integrate, we get a constant of integration, which could be any real value. This means **we don't know what the exact original function was**.



The curve with equation $y = f(x)$ passes through $(1,3)$. Given that $f'(x) = 3x^2$, find the equation of the curve.

?

Test Your Understanding

Edexcel C1 May 2014 Q10

A curve with equation $y = f(x)$ passes through the point (4, 25).

Given that

$$f'(x) = \frac{3}{8}x^2 - 10x^{-\frac{1}{2}} + 1, \qquad x > 0$$

(a) find $f(x)$, simplifying each term.

(5)

(b) Find an equation of the normal to the curve at the point (4, 25).

Give your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(5)

To keep you occupied if you finish (a) quickly!

? a

? b

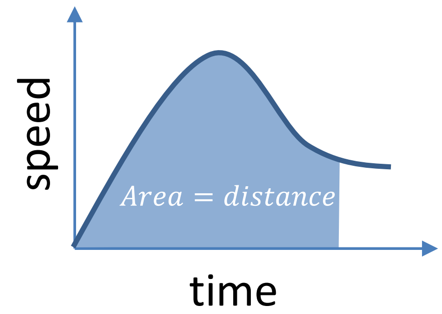
Exercise 13C

Pearson Pure Mathematics Year 1/AS
Pages 294-295

Definite Integration

So far we've seen integration as '**the opposite of differentiation**', allowing us to find $y = f(x)$ when we know the gradient function $y = f'(x)$.

In practical settings however the most useful use of integration is that **it finds the area under a graph**. Remember at GCSE for example when you estimated the area under a speed-time graph, using trapeziums, to get the distance?
If you knew the equation of the curve, you could get the exact area!

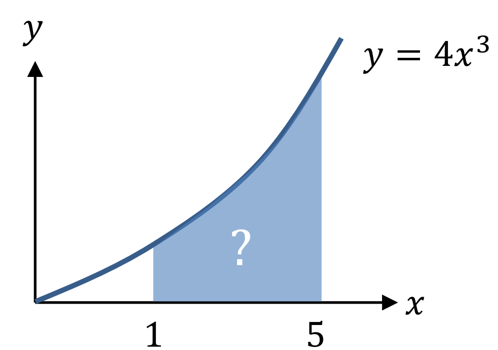


Before we do this, we need to understand how to find a **definite integral**:

These are known as **limits**, which give the values of x we're finding the area between.

We integrate as normal, but put expression in **square brackets**, meaning **we still need to evaluate the integrated expression using the limits**.

$$\int_1^5 4x^3\,dx = [x^4]_1^5$$
$$= (5^4) - (1^4)$$
$$= 624$$



Write $(\dots) - (\dots)$ and evaluate the expression for each of the limits, top one first.

Another Example

$$\int_{-3}^{3} x^2 + 1\, dx = \boxed{?}$$

$$= \boxed{?}$$

$$= \boxed{?}$$

$$=$$

We **DON'T have a constant of integration** when doing definite integration. I'll explain why later.

Write out you working EXACTLY as seen here. The $(\ldots) - (\ldots)$ brackets are particularly crucial as you'll otherwise likely make a sign error.

"Use of Technology" Monkey says:

You can use the $\left[\!\left[\int_b^a \square\right]\!\right]$ button on your calculator to evaluate definite integrals.

<u>But only use it to check your answer.</u>

Problem Solving

Given that P is a constant and $\int_1^5 (2Px + 7)\,dx = 4P^2$, show that there are two possible values for P and find these values.

?

Remember: P is a constant, so just treat it as a number.

Exercise 13D

Pearson Pure Mathematics Year 1/AS
Page 297

(Classes in a rush may want to skip this exercise and go to the next section, which continues definite integration, but in the context of areas under graphs).

Extension

1 *[MAT 2009 1A]* The smallest value of

$$I(a) = \int_0^1 (x^2 - a)^2 \, dx$$

as a varies, is what?

?

2 *[MAT 2015 1D]* Let

$f(x) = \int_0^1 (xt)^2 \, dt$ and $g(x) = \int_0^x t^2 \, dt$

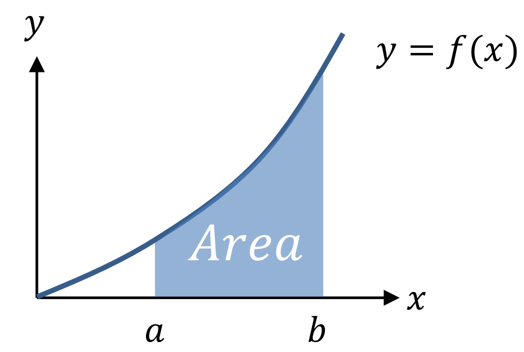
Let $A > 0$. Which of the following statements are true?

A) $g\big(f(A)\big)$ is always bigger than $f\big(g(A)\big)$
B) $f\big(g(A)\big)$ is always bigger than $g\big(f(A)\big)$
C) They are always equal.
D) $f\big(g(A)\big)$ is bigger if $A < 1$, and $g\big(f(A)\big)$ is bigger if $A > 1$.
E) $g\big(f(A)\big)$ is bigger if $A < 1$, and $f\big(g(A)\big)$ is bigger if $A > 1$.

?

Areas under curves

Earlier we saw that the definite integral $\int_b^a \boldsymbol{f(x)}\ \boldsymbol{dx}$ gives the **area** between a positive curve $y = f(x)$, the **x-axis**, and the lines $x = a$ and $x = b$.
(We'll see <u>why</u> this works in a sec)



Find the area of the finite region between the curve with equation $y = 20 - x - x^2$ and the x-axis.

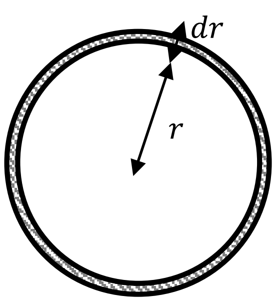
?

Just for your interest...

Why does integrating a function give you the area under the graph?

Part 1:
You're already familiar with the idea that gradient is the rate at which a quantity changes, and we consider an infinitesimally small change in the variable.
You could consider the gradient as the little bit you're adding on each time.

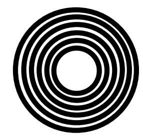
Here's some practical examples using formulae you covered in your younger years!



If you wanted to consider the rate at which the area of a circle increases with respect to the radius, consider a small change dr in the radius. The change in area is an infinitely thin ring, which looks like you've drawn a circumference.
So what is this rate?

$$A = \pi r^2$$
$$\therefore \frac{dA}{dr} = 2\pi r$$

OMG THAT <u>IS</u> THE CIRCUMFERENCE

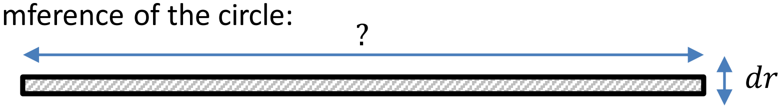


So to draw a shaded circle (which has area!), it's a bit like repeatedly drawing the circumference of a circle with gradually increasing radius. **Since the circumference is what the <u>area is increasing by each time</u>, the circumference is the gradient of the area of the circle, and conversely (by the definition of integration being the opposite of differentiation), the area is the integral of the circumference.**

<u>You might be rightly upset that you can't add a length to an area</u> (only an area to an area, innit).
But by considering the infinitely thin width dr of the circumference you're drawing, it does have area!

$$\frac{dA}{dr} = 2\pi r \quad \rightarrow \quad dA = 2\pi r\, dr$$

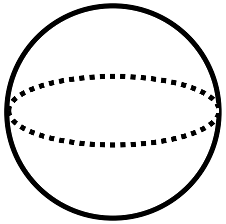
i.e. the change in area, dA, is $2\pi r \times dr$
If we 'roll out' the added area we're adding each time, this forms a rectangle, whose length hopefully corresponds to the circumference of the circle:



If the added area is $2\pi r\, dr$ and the thickness is dr, then the length is $\frac{2\pi r\, dr}{dr} = 2\pi r$ as expected.

Part 2:

It gets better!



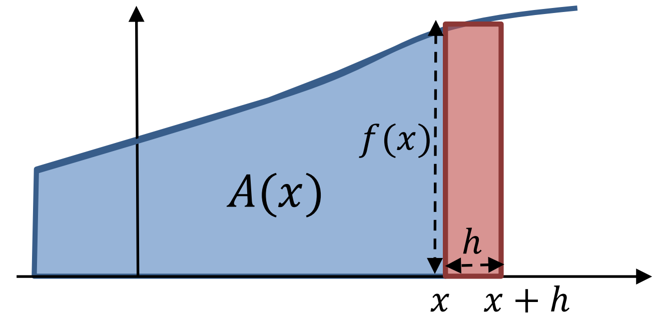
Consider the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

OMG THAT'S THE SURFACE AREA

This works for a similar reason to before.



And the same principle applies to area under a graph. If $A(x)$ gives the area up to x, then what we're 'adding on each time' (i.e. the gradient) is sort of like drawing a line of length $f(x)$ at the right-most end each time (or more accurately, an infinitely thin rectangle of area $f(x) \times h$). Thus if $f(x)$ is the gradient of the area, then conversely the area is the integral of $f(x)$.

This gives us an **intuitive** sense of why it works, but we need a formal proof:

Using the diagram above, the area up to $x + h$, i.e. $A(x+h)$, is approximately the area up to x plus the added rectangular strip:

$$A(x+h) \approx A(x) + (f(x) \times h)$$

Rearranging:

$$f(x) \approx \frac{A(x+h) - A(x)}{h}$$

In the limit, as the rectangle becomes infinitely thin, this becomes an equality:

$$f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

But we know from differentiating by first principles that:

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

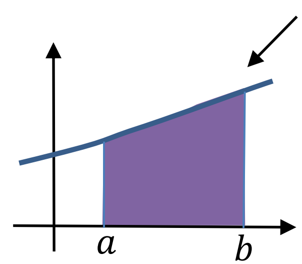
$$\therefore f(x) = A'(x)$$

And thus we have proven that the gradient of the area function is indeed $f(x)$ (and hence the integral of $f(x)$ the area).

But we're missing one final bit: Why does $\int_b^a f(x)\,dx$ give the area between $x = a$ and $x = b$?

Since $f(x) = A'(x)$, the area function $A(x)$ is the integral of $f(x)$. Thus:

$$\int_b^a f(x)\,dx = [A(x)]_a^b \quad = A(b) - A(a)$$



The area between a and b is the area up to b, i.e. $A(b)$, with the area up to a, i.e. $A(a)$, cut out. This gives $A(b) - A(a)$ as expected. Because we obtained $A(x)$ by an integration, we have a constant of integration $+c$. But because of the subtraction $A(b) - A(a)$, these constants in each of $A(b)$ and $A(a)$ cancel, which explains why the constant is not present in definite integration.

This is known as the Fundamental Theorem of Calculus.

Test Your Understanding

Edexcel C2 Jan 2013 Q9c

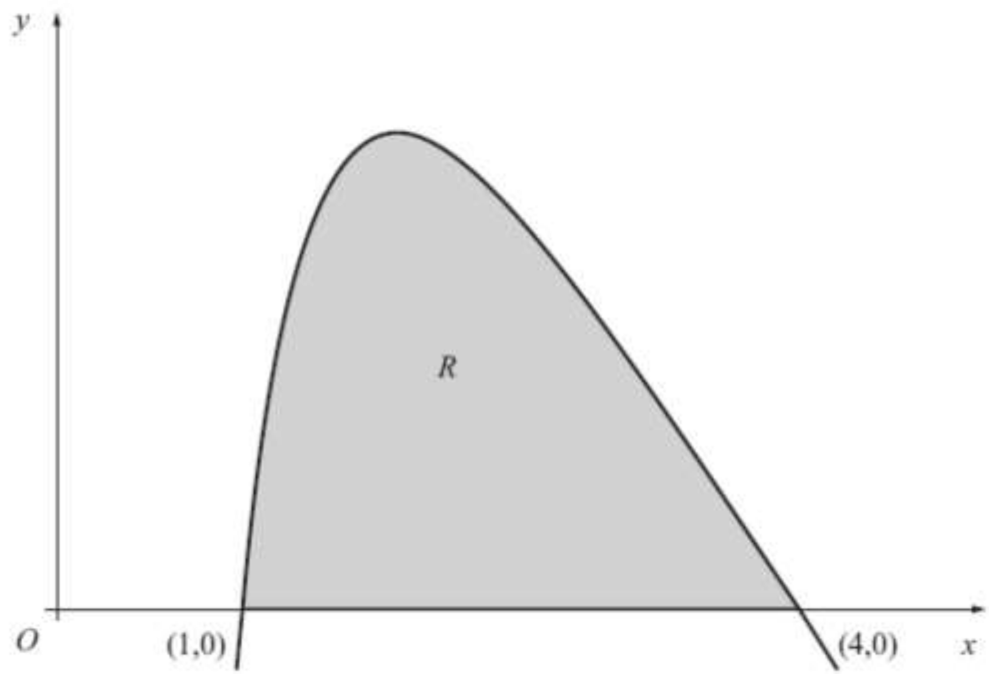


Figure 2

The finite region R, as shown in Figure 2, is bounded by the x-axis and the curve with equation

$$y = 27 - 2x - 9\sqrt{x} - \frac{16}{x^2}, \quad x > 0.$$

The curve crosses the x-axis at the points (1, 0) and (4, 0).

(*c*) Use integration to find the exact value for the area of R.

(6)

?

Exercise 13E

Pearson Pure Mathematics Year 1/AS
Pages 299-300

Extension

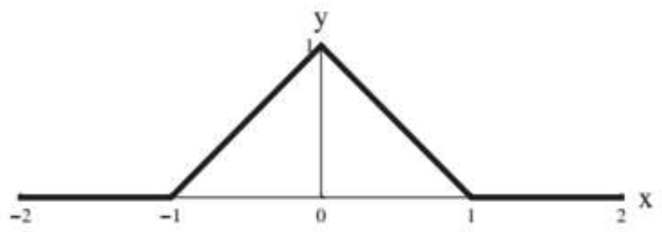
1 *[MAT 2007 1H]* Given a function $f(x)$, you are told that

$$\int_0^1 3f(x)\,dx + \int_1^2 2f(x)\,dx = 7$$
$$\int_0^2 f(x)\,dx + \int_1^2 f(x)\,dx = 1$$

It follows that $\int_0^2 f(x)\ dx$ equals what?

?

2 *[MAT 2011 1G]*
A graph of the function $y = f(x)$ is sketched on the axes below:



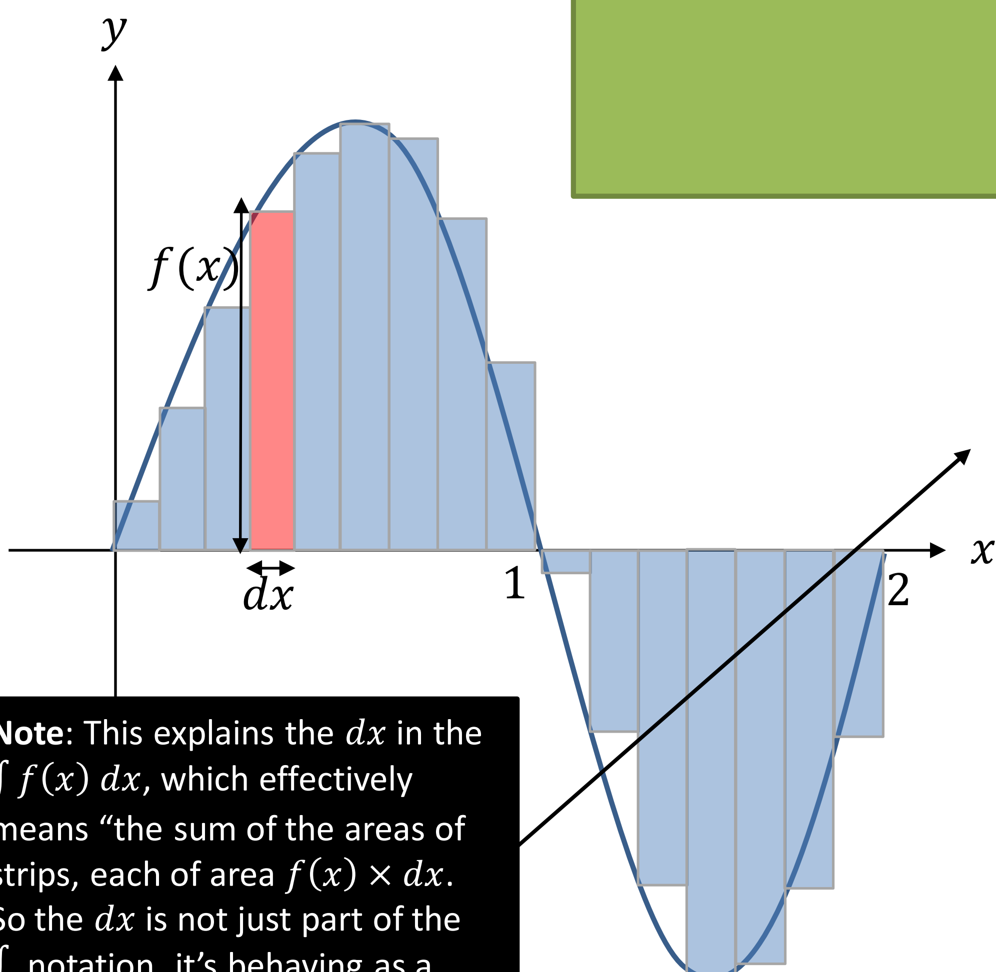
What is the value of $\int_{-1}^{1} f(x^2-1)\ dx$?

?

‘Negative Areas’

Sketch the curve $y = x(x-1)(x-2)$ (which expands to give $y = x^3 - 3x^2 + 2x$).
Now calculate $\int_0^2 x(x-1)(x-2)\ dx$. Why is this result surprising?

?



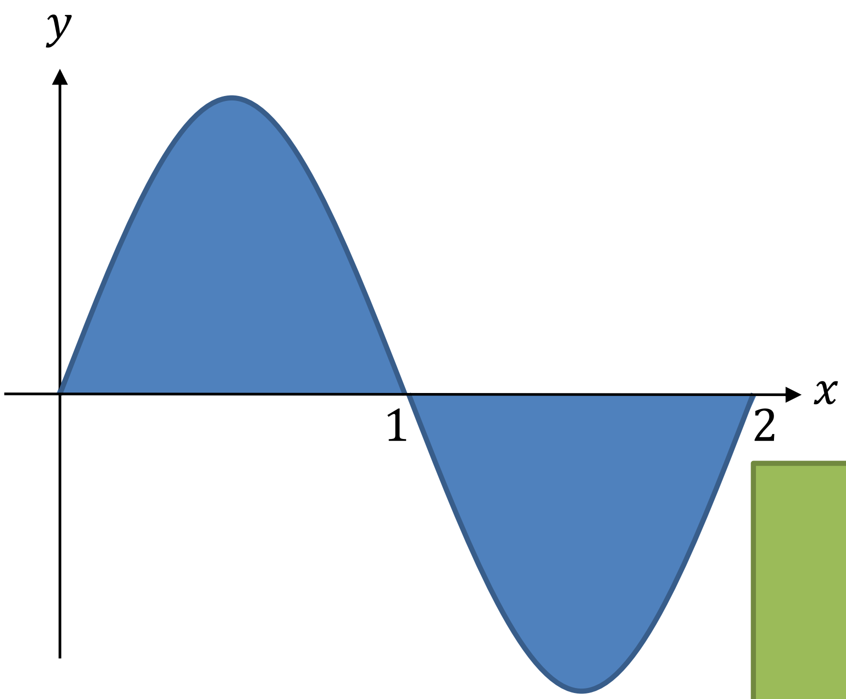
Integration $\int f(x)\ dx$ is just the sum of areas of infinitely thin rectangles, where the current y value (i.e. $f(x)$) is each height, and the widths are dx.
i.e. The area of each is $f(x) \times dx$

The problem is, when $f(x)$ is negative, then $f(x) \times dx$ is negative, i.e. a negative area!
The result is that the ‘positive area’ from 0 to 1 is cancelled out by the ‘negative area’ from 1 to 2, giving an overall ‘area’ of 0.
So how do we resolve this?

Note: This explains the dx in the $\int f(x)\ dx$, which effectively means “the sum of the areas of strips, each of area $f(x) \times dx$. So the dx is not just part of the $\int$ notation, it’s behaving as a physical quantity! (i.e. length)

Example

Find the total area bound between the curve $y = x(x-1)(x-2)$ and the x-axis.



Strategy:

?

? Solution

Test Your Understanding

Edexcel C2 May 2013 Q6

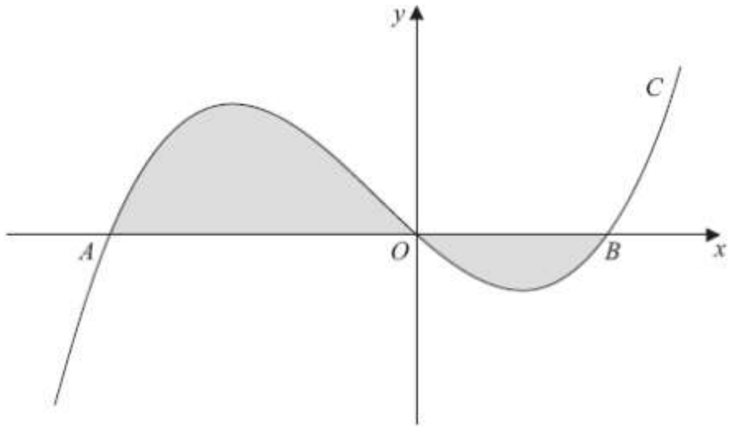


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = x(x + 4)(x - 2).$$

The curve C crosses the x-axis at the origin O and at the points A and B.

(*a*) Write down the x-coordinates of the points A and B.
(1)

The finite region, shown shaded in Figure 3, is bounded by the curve C and the x-axis.

(*b*) Use integration to find the total area of the finite region shown shaded in Figure 3.
(7)

(a) ?

(b) ?

Exercise 13F

Pearson Pure Mathematics Year 1/AS
Pages 301-302

Hint: It's not actually even possible to integrate 2^{x^2}, but we can still sketch the graph. Reflect on what $\frac{dI}{da}$ actually means.

Extension

1 *[MAT 2010 1I]* For a positive number a, let

$$I(a) = \int_0^a \left(4 - 2^{x^2}\right) dx$$

Then $\frac{dI}{da} = 0$ when a is what value?

?

2 *[STEP I 2014 Q3]*
The numbers a and b, where $b > a \geq 0$, are such that

$$\int_a^b x^2\, dx = \left(\int_a^b x\, dx\right)^2$$

(i) In the case $a = 0$ and $b > 0$, find the value of b.

(ii) In the case $a = 1$, show that b satisfies

$$3b^3 - b^2 - 7b - 7 = 0$$

Show further, with the help of a sketch, that there is only one (real) value of b that satisfies the equation and that it lies between 2 and 3.

(iii) Show that $3p^2 + q^2 = 3p^2 q$, where $p = b + a$ and $q = b - a$, and express p^2 in terms of q.
Deduce that $1 < b - a \leq \frac{4}{3}$

Guidance for this problem on next slide.

Guidance for Extension Problem 2

[STEP I 2014 Q3] The numbers a and b, where $b > a \geq 0$, are such that $\int_a^b x^2\,dx = \left(\int_a^b x\,dx\right)^2$

(i) In the case $a = 0$ and $b > 0$, find the value of b.

(ii) In the case $a = 1$, show that b satisfies

$$3b^3 - b^2 - 7b - 7 = 0$$

Show further, with the help of a sketch, that there is only one (real) value of b that satisfies the equation and that it lies between 2 and 3.

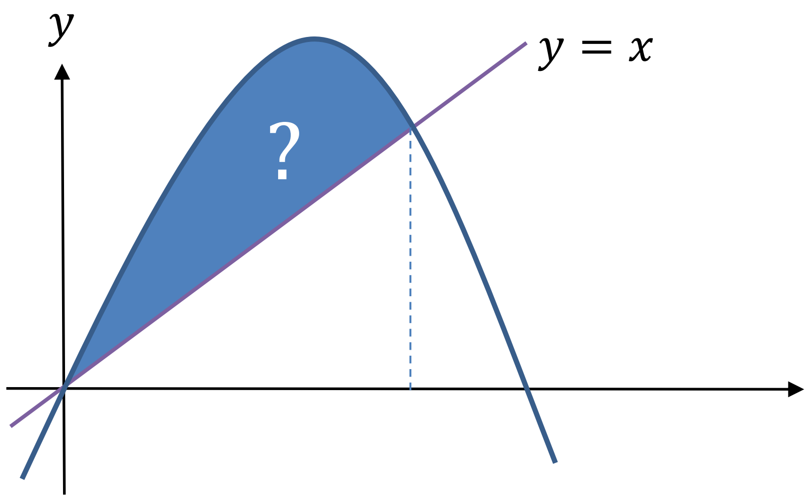
(iii) Show that $3p^2 + q^2 = 3p^2q$, where $p = b + a$ and $q = b - a$, and express p^2 in terms of q. Deduce that $1 < b - a \leq \frac{4}{3}$

This question was actually devised to address what happens when students misunderstand or mis-apply a "rule" of mathematics and *it turns out to give the right answer*. Part (i) starts you off gently: integrating both terms, squaring the RHS and solving very quickly gives $b = \frac{4}{3}$. Part (ii) develops in much the same way, but with a non-zero lower limit to the integrals, and we immediately see that the algebra gets much more involved. Importantly, it should be very clear that whatever expression materialises must have $(b - 1)$ as a factor (since setting $b = a$ would definitely give a zero area, thus trivially satisfying the given integral statement). This leads to the required cubic equation.

The final part of (ii) requires a mixture of different ideas (and can be done in a number of different ways). The most basic approach to demonstrating that a cubic curve has only one zero is to illustrate that both of its TPs lie on the same side of the x-axis (or to show there are no TPs). The popular *Change-of-Sign Rule* for continuous functions can be used to identify the position of this zero.

Having got you started with some simple lower limits, part (iii) develops matters more generally, and derives the (perhaps) surprising result that the exploration of this initial "stupid idea" requires b and a to be "not too far apart" to an extent that is easily identifiable.

Areas between curves and lines



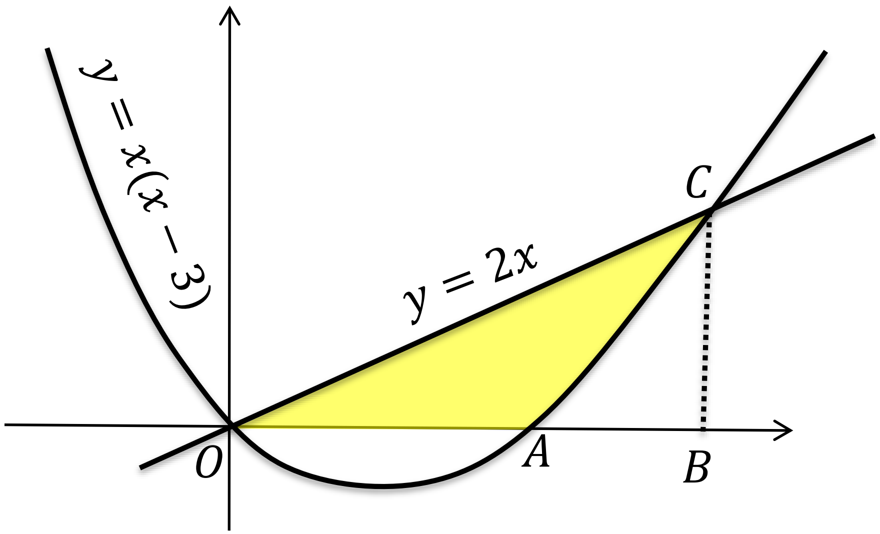
How could we find the area between the line and the curve?

?

Determine the area between the lines with equations $y = x(4 - x)$ and $y = x$

?

A Harder One



[Textbook] The diagram shows a sketch of the curve with equation $y = x(x-3)$ and the line with equation $y = 2x$. Find the area of the shaded region OAC.

What areas should we subtract this time?

?

?

Test Your Understanding

Edexcel C2 May 2012 Q5

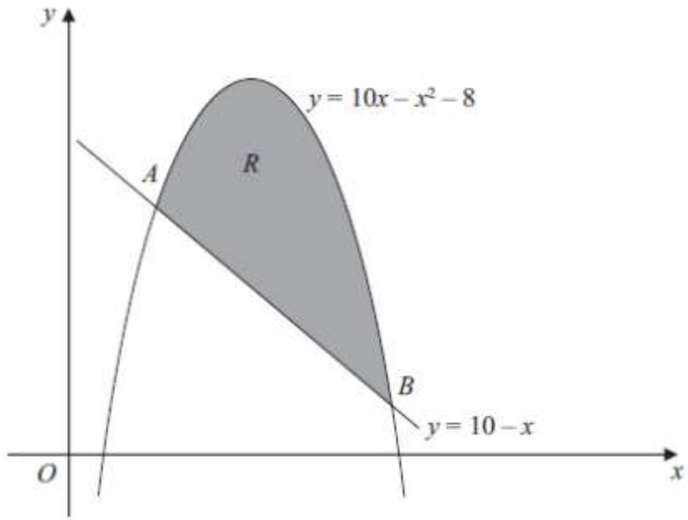


Figure 2 shows the line with equation $y = 10 - x$ and the curve with equation $y = 10x - x^2 - 8$.

The line and the curve intersect at the points A and B, and O is the origin.

(*a*) Calculate the coordinates of A and the coordinates of B.

(5)

The shaded area R is bounded by the line and the curve, as shown in Figure 2.

(*b*) Calculate the exact area of R.

(7)

a ? a

b ? b

Alternative Method:
If the top curve has equation $y = f(x)$ and the bottom curve $y = g(x)$, the area between them is:

$$\int_b^a \left(f(x) - g(x)\right) dx$$

This means you can integrate a single expression to get the final area, without any adjustment required after.

Exercise 13G

Pearson Pure Mathematics Year 1/AS
Pages 304-306

Extension

1 *[MAT 2005 1A]* What is the area of the region bounded by the curves $y = x^2$ and $y = x + 2$?

? 1

2 *[MAT 2016 1H]* Consider two functions

$$f(x) = a - x^2$$
$$g(x) = x^4 - a$$

For precisely which values of $a > 0$ is the area of the region bounded by the x-axis and the curve $y = f(x)$ bigger than the area of the region bounded by the x-axis and the curve $y = g(x)$?
(Your answer should be an inequality in terms of a)

? 2