P1 Chapter 9 :: Trigonometric Ratios

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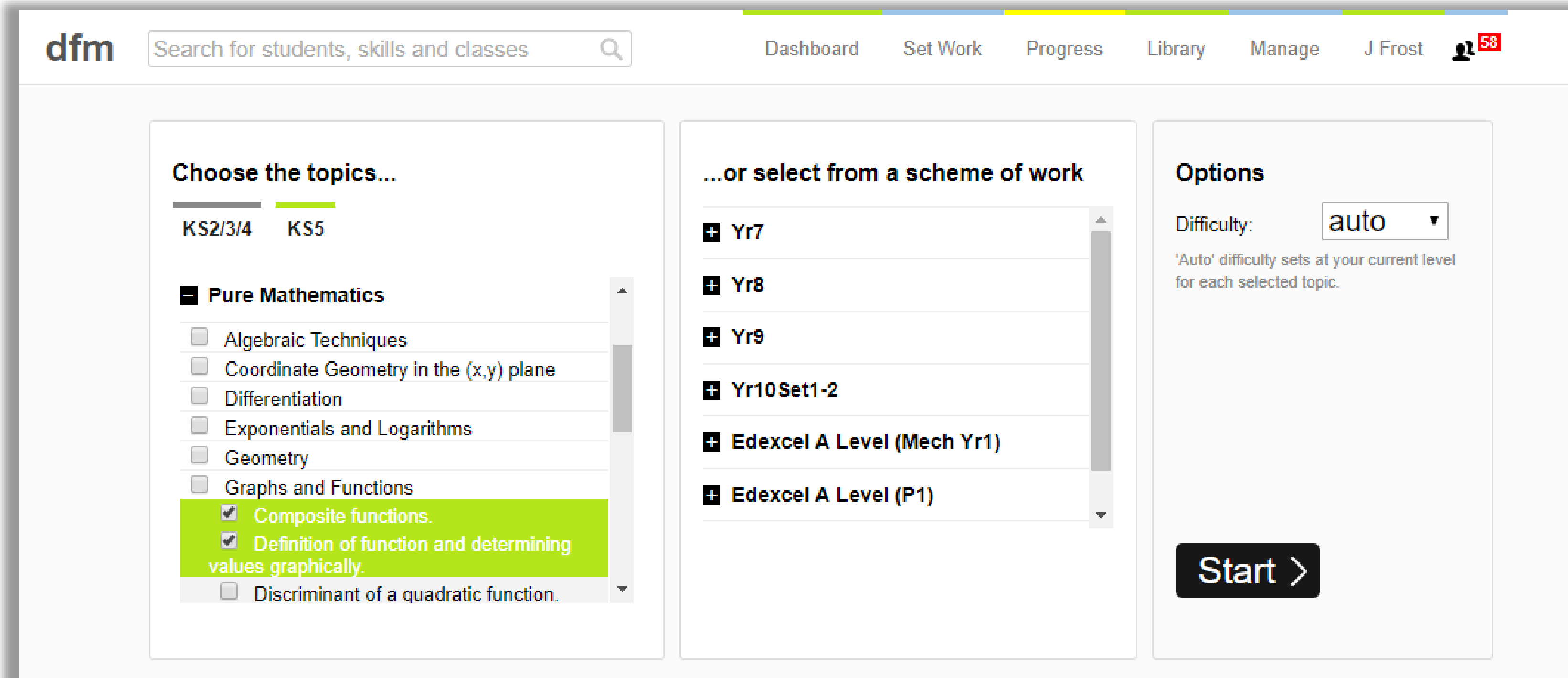

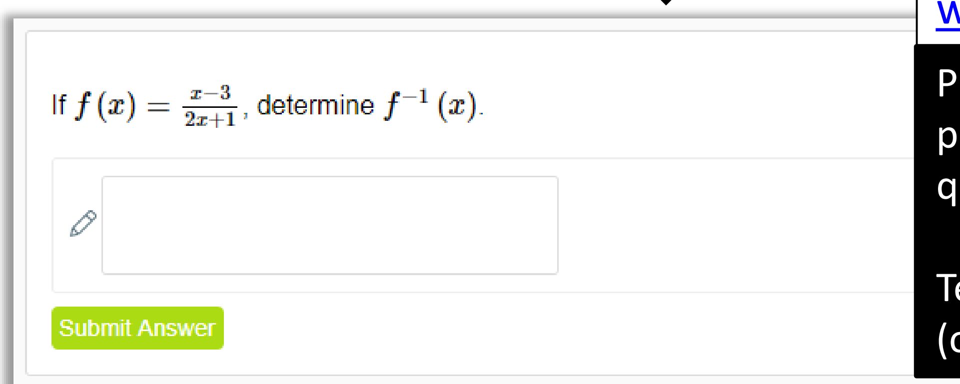

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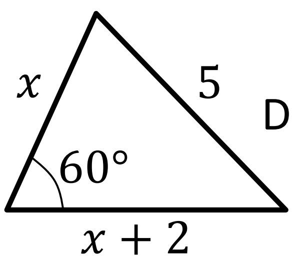
Chapter Overview

There is technically no new content in this chapter since GCSE.
However, the problems might be more involved than at GCSE level.

1:: Sine/Cosine Rule



Determine x.

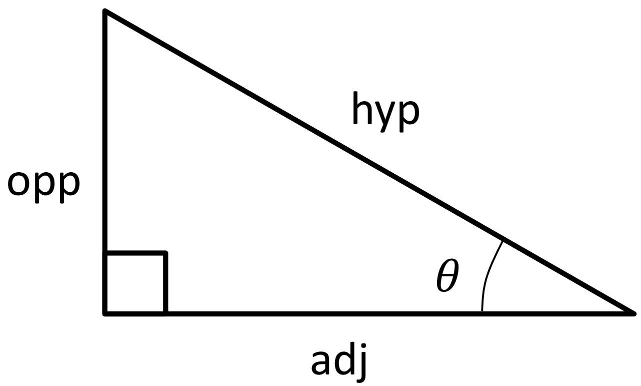
2:: Areas of Triangles

In ΔABC, $AB = 5, BC = 6$ and $\angle ABC = x$. Given that the area of $\angle ABC$ is 12cm^2 and that AC is the longest side, find the value of x.

3:: Graphs of Sine/Cosine/Tangent

Sketch $y = \sin(2x)$ for $0 \leq x \leq 360°$

RECAP :: Right-Angled Trigonometry

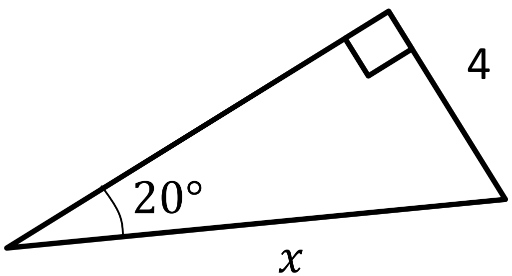


You are probably familiar with the formula: $\sin(\theta) = \frac{opp}{hyp}$

But what is the *conceptual* definition of sin ?

?

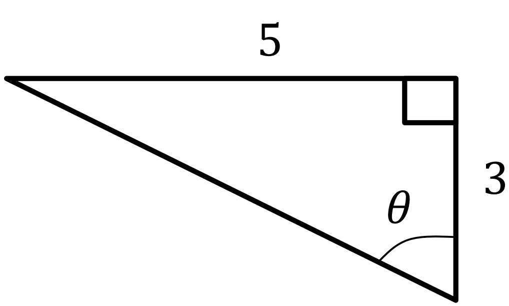
Remember that a ratio just means the 'relative size' between quantities (in this case lengths). For this reason, sin/cos/tan are known as "trigonometric ratios".



Find x.

?

Tip: You can swap the thing you're dividing by and the result. e.g. $\frac{8}{2} = 4 \rightarrow \frac{8}{4} = 2$. I call this the '*swapsie trick*'.



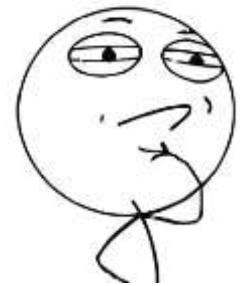
Find θ.

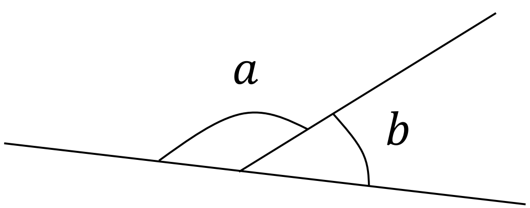
?

You may have been taught "use tan^{-1} whenever you're finding an angle", and therefore write the second line directly. This is fine, but I prefer to always write the first line, then see the problem as a 'changing the subject' one. We need to remove the tan on front of the θ, so apply tan^{-1} to each side of the equation to 'cancel out' the tan on the LHS.

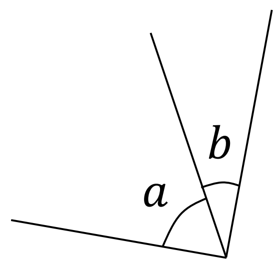
Just for your interest…

Have you ever wondered why "cosine" contains the word "sine"?

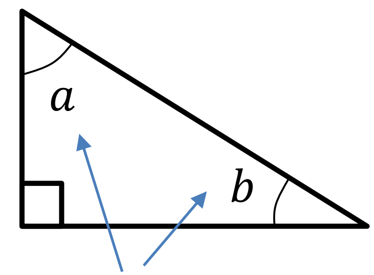




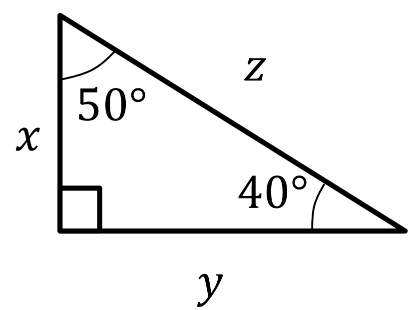
Supplementary Angles
add to 180°



Complementary Angles
add to 90°



Therefore these angles are complementary.



$\cos(50) =$?

$\sin(40) =$?

$\therefore \mathbf{\cos(50) = \sin(40)}$

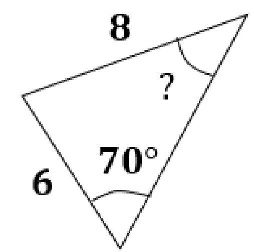
i.e. The **cosine** of an angle is the **sine** of the **complementary** angle.
Hence **cosine = COMPLEMENTARY SINE**

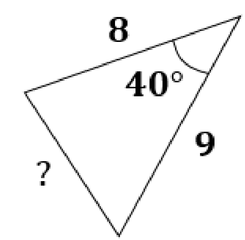
ME-WOW!

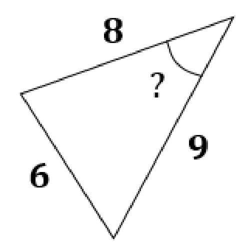
OVERVIEW: Finding missing sides and angles

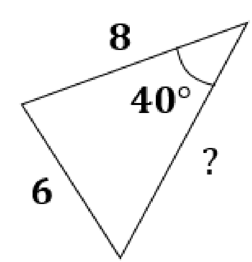
When triangles are not right-angled, we can no longer use simple trigonometric ratios, and must use the cosine and sine rules.

You have	You want	Use
#1: Two angle-side opposite pairs	Missing angle or side in one pair	?
#2 Two sides known and a missing side opposite a known angle	Remaining side	?
#3 All three sides	An angle	?
#4 Two sides known and a missing side <u>not</u> opposite known angle	Remaining side	?



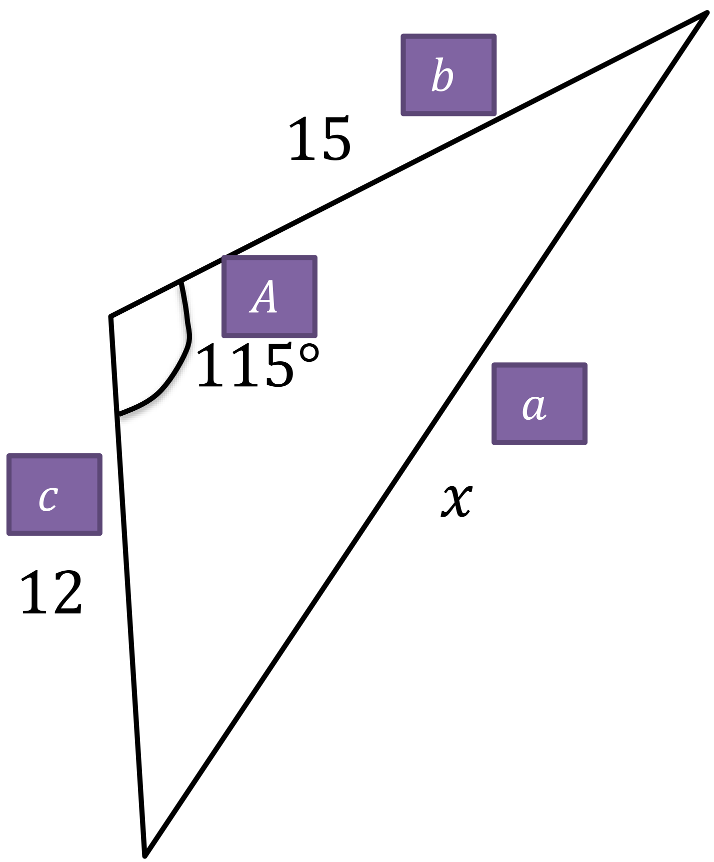






Cosine Rule

We use the **cosine rule** whenever we have **three sides** (and an angle) involved.



Proof at end of PowerPoint.

Cosine Rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

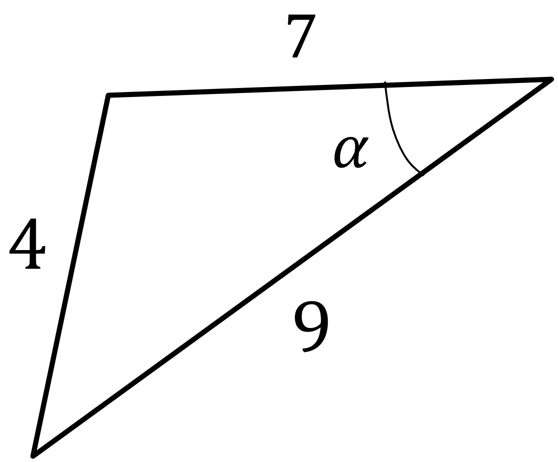
How are sides labelled ?

Calculation?

Dealing with Missing Angles

You have	You want	Use
All three sides	An angle	Cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$



Tip: The brackets are not needed, but students who forget about BIDMAS see $4^2 = (7^2 + 9^2 - 2 \times 7 \times 9)\cos\alpha$ and hence incorrectly simplify to $16 = 4\cos\alpha$

?

?

?

?

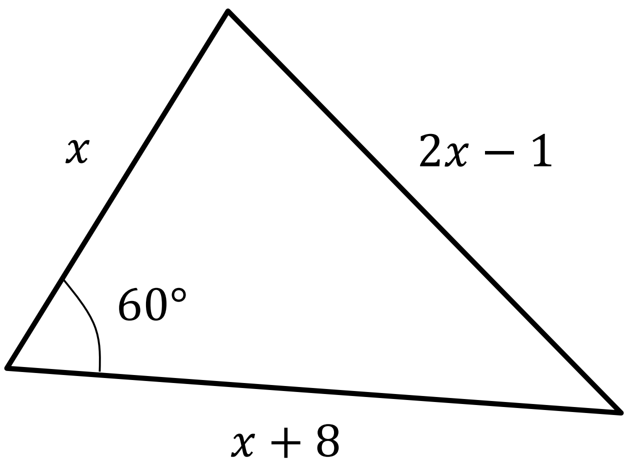
Label sides then substitute into formula.

Simplify each bit of formula.

Rearrange (I use 'subtraction swapsie trick' to swap thing you're subtracting and result)

Textbook Note: The textbook presents the rearrangement of the cosine rule: $\cos A = \frac{b^2+c^2-a^2}{2bc}$ to find missing angles. I'd personally advise against using this as: (a) It's another formula to remember. (b) Anything that gives you less practice of manipulating/rearranging equations is probably a bad thing. (c) You won't get to use the swapsie trick. ☹

Harder Ones



Determine the value of x.

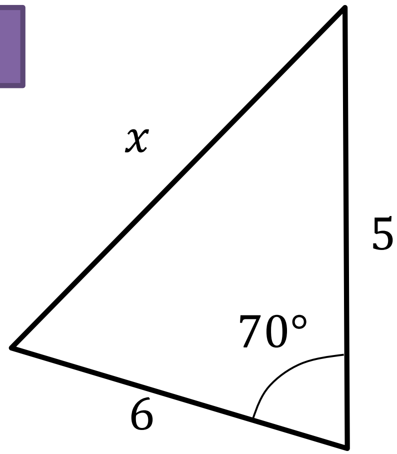
?

[From textbook] Coastguard station B is 8 km, on a bearing of $060°$, from coastguard station A. A ship C is 4.8 km on a bearing of $018°$, away from A. Calculate how far C is from B.

?

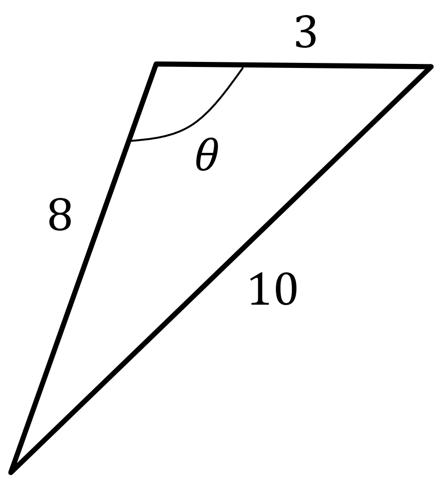
Test Your Understanding

1



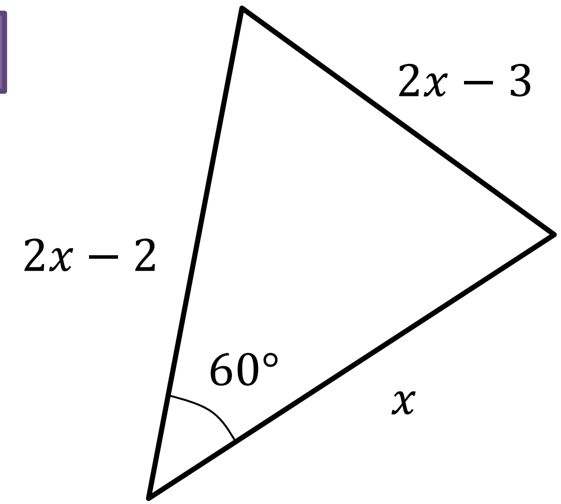
$x =$?

2



$\theta =$?

3



$x =$?

Fro Note: You will get an obtuse angle whenever you inverse cos a negative value.

Exercise 9A

Pearson Pure Mathematics Year 1/AS
Pages 177-179

Extension

1 *[STEP I 2009 Q4i]*
The sides of a triangle have lengths $p-q$, p and $p+q$, where $p>q>0$. The largest and smallest angles of the triangle are α and β respectively. Show by means of the cosine rule that

$$4(1-\cos\alpha)(1-\cos\beta)=\cos\alpha+\cos\beta$$

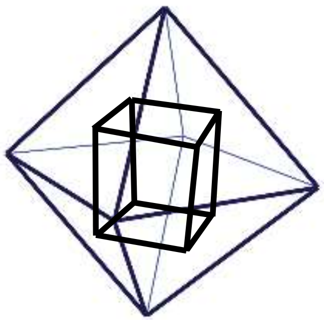
?

2 [STEP I 2007 Q5] Note: a regular octahedron is a polyhedron with eight faces each of which is an equilateral triangle.

(i) Show that the angle between any two faces of a regular octahedron is $\arccos\left(-\frac{1}{3}\right)$

(ii) Find the ratio of the volume of a regular octahedron to the volume of the cube whose vertices are the centres of the faces of the octahedron.

Solutions for Q2 on next slide.



Solutions to Extension Question 2

[STEP I 2007 Q5] Note: a regular octahedron is a polyhedron with eight faces each of which is an equilateral triangle.

(i) Show that the angle between any two faces of a regular octahedron is $\arccos\left(-\frac{1}{3}\right)$

(ii) Find the ratio of the volume of a regular octahedron to the volume of the cube whose vertices are the centres of the faces of the octahedron.

(Official solutions) Big, clear diagram essential!

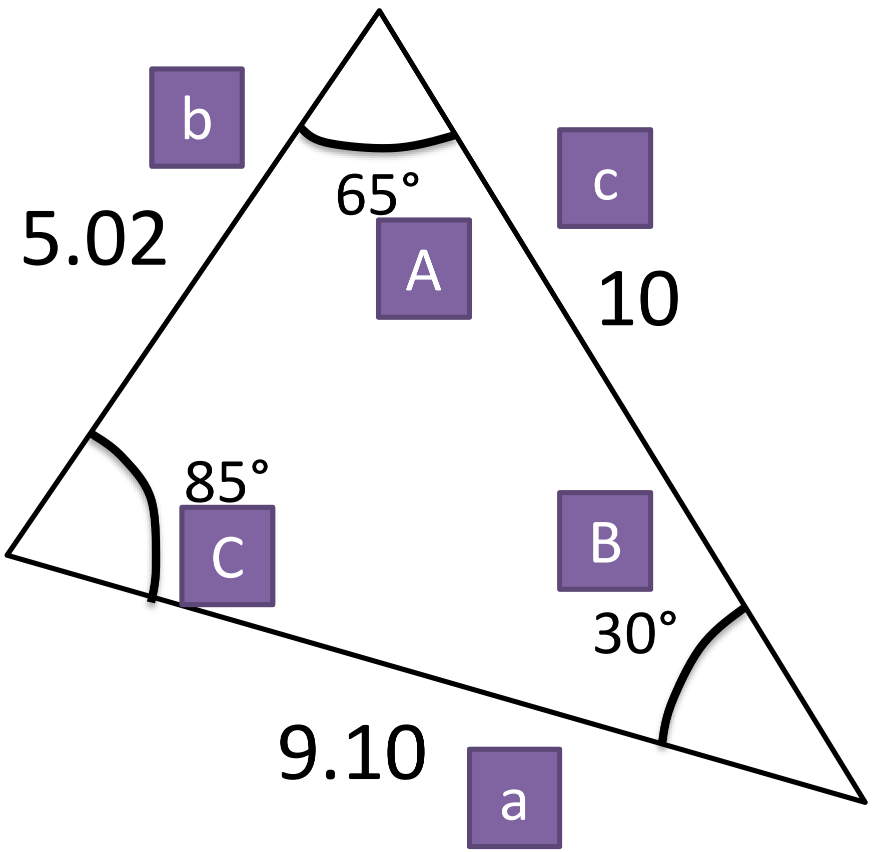
(i)

?

(ii)

?

The Sine Rule

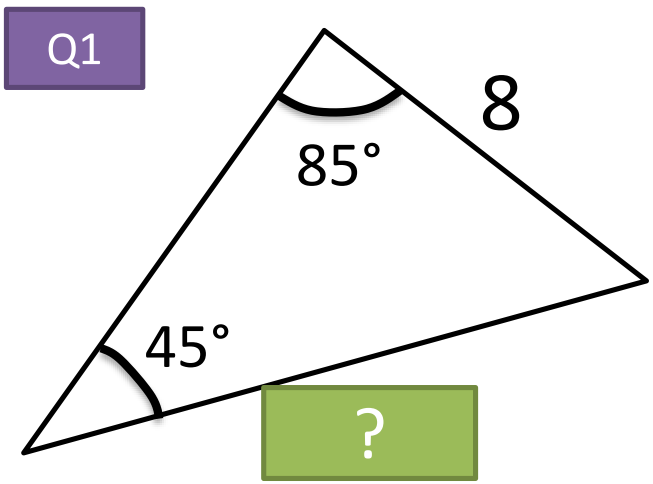


For this triangle, try calculating each side divided by the sin of its opposite angle. What do you notice in all three cases?

?

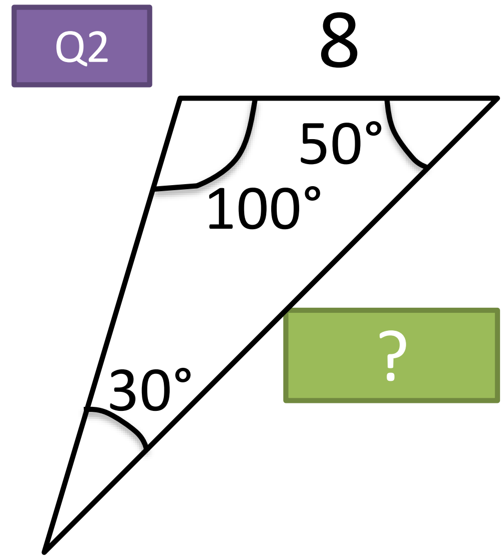
You have	You want	Use
#1: Two angle-side opposite pairs	Missing angle or side in one pair	Sine rule

Examples



$$\frac{x}{\sin 85} = \frac{8}{\sin 45}$$

$$x = \frac{8 \sin 85}{\sin 45} = 11.27$$



$$\frac{x}{\sin 100} = \frac{8}{\sin 30}$$

$$x = \frac{8 \sin 100}{\sin 30} = 15.76$$

You have	You want	Use
#1: Two angle-side opposite pairs	Missing angle or side in one pair	Sine rule

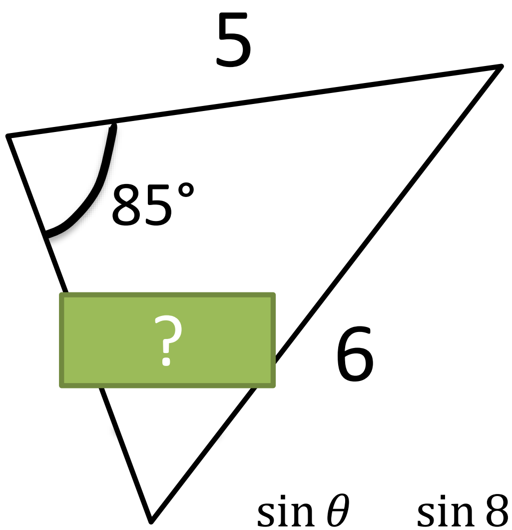
Examples

When you have a missing angle, it's better to reciprocate to get:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

i.e. in general put the missing value in the numerator.

Q3



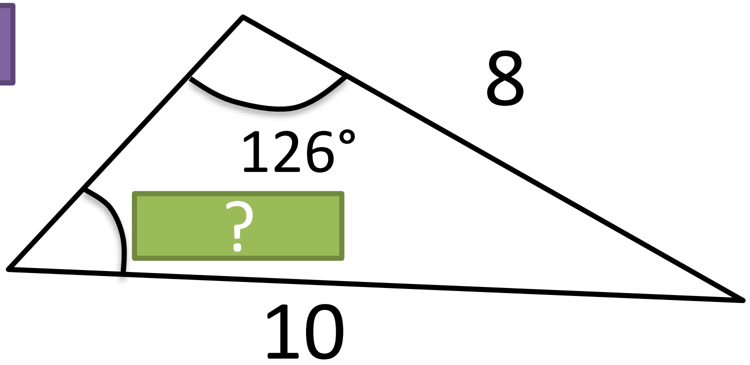
$$\frac{\sin\theta}{5} = \frac{\sin 85}{6}$$

$$\sin\theta = \frac{5\sin 85}{6}$$

$$\theta = \sin^{-1}\left(\frac{5\sin 85}{6}\right)$$

$$= 56.11°$$

Q4



$$\frac{\sin\theta}{8} = \frac{\sin 126°}{10}$$

$$\sin\theta = \frac{8\sin 126}{10}$$

$$\theta = \sin^{-1}\left(\frac{8\sin 126}{10}\right)$$

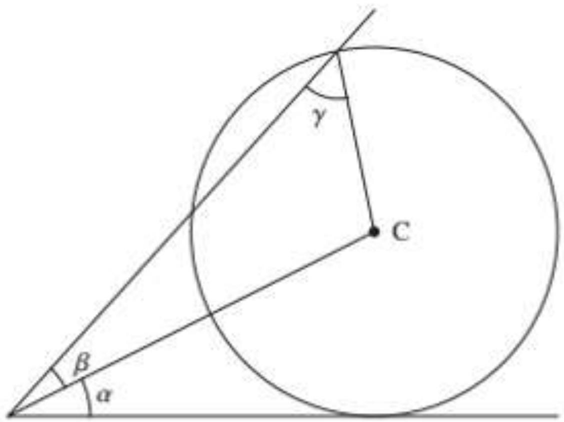
$$= 40.33°$$

Exercise 9B

Pearson Pure Mathematics Year 1/AS
Pages 181-183

Extension

1 *[MAT 2011 1E]*
The circle in the diagram has centre C. Three angles α, β, γ are also indicated.



The angles α, β, γ are related by the equation:

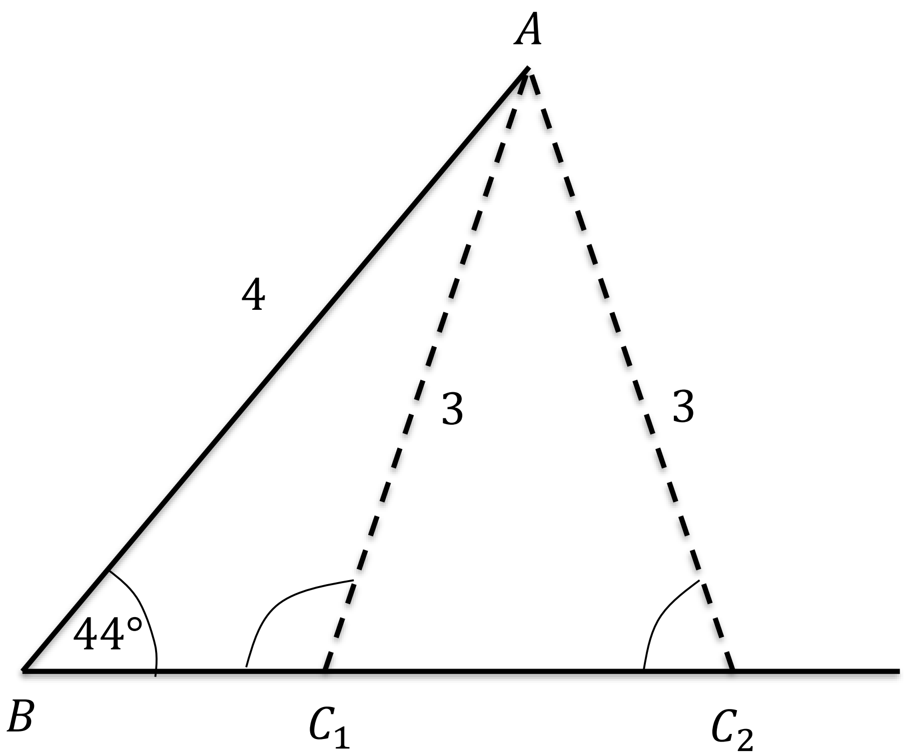
A) $\cos\alpha = \sin(\beta + \gamma)$
B) $\sin\beta = \sin\alpha \sin\gamma$
C) $\sin\beta(1 - \cos\alpha) = \sin\gamma$
D) $\sin(\alpha + \beta) = \cos\gamma \sin\alpha$

?

The 'Ambiguous Case'

Construct triangle ABC such that $AB = 4$, $AC = 3$ and $\angle ABC = 44°$.

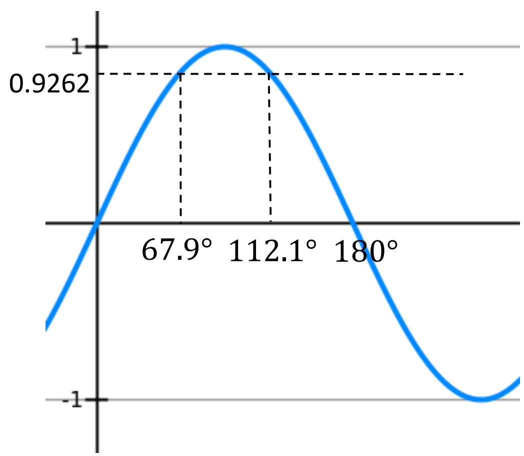
The 'Ambiguous Case'



Suppose you are told that $AB = 4$, $AC = 3$ and $\angle ABC = 44°$. What are the possible values of $\angle ACB$?

C is somewhere on the horizontal line. There's two ways in which the length could be 3. Using the sine rule:

$$\frac{\sin C}{4} = \frac{\sin 44}{3}$$
$$C = \sin^{-1}(0.9262)$$



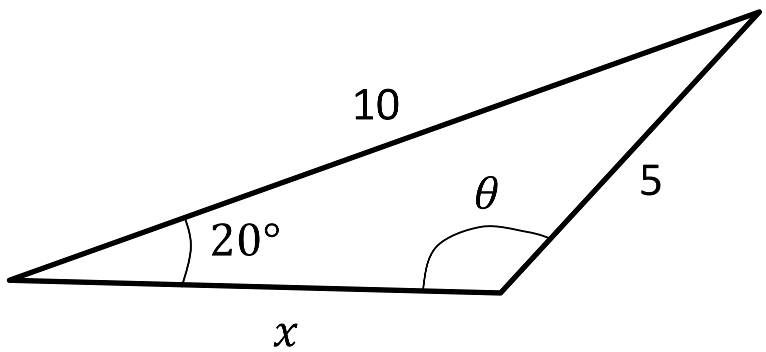
Your calculator will give the acute angle of 67.9° (i.e. C_2). But if we look at a graph of sin, we can see there's actually a second value for $\sin^{-1}(0.9262)$, corresponding to angle C_1.

✏ The sine rule produces two possible solutions for a missing angle:

$$\sin\theta = \sin(180° - \theta)$$

Whether we use the acute or obtuse angle depends on context.

Test Your Understanding



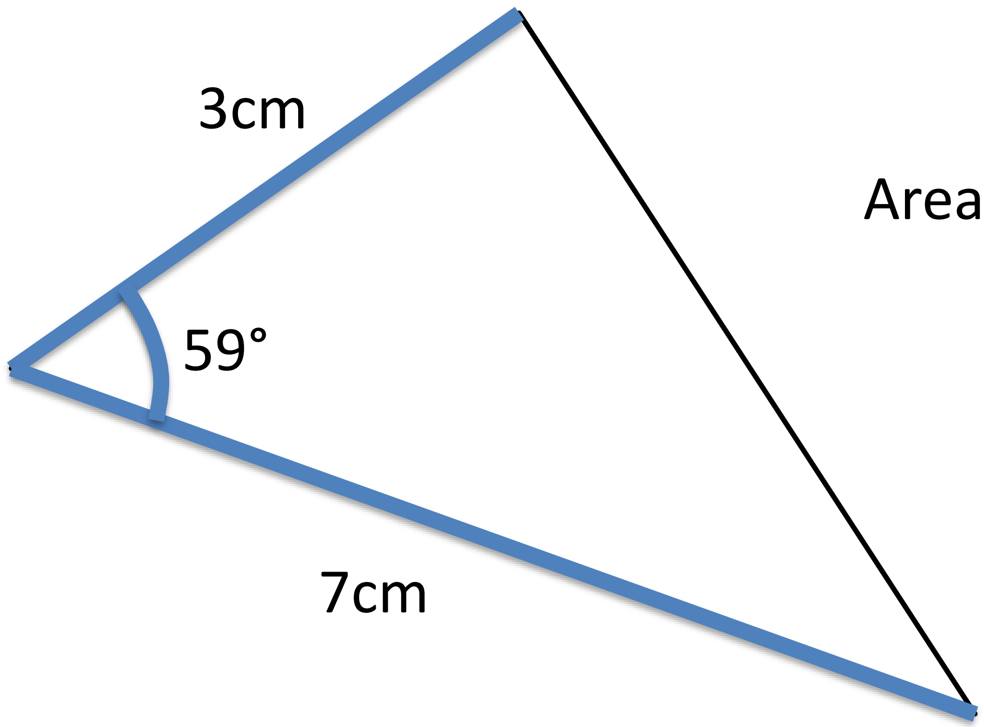
Given that the angle θ is obtuse, determine θ and hence determine the length of x.

?

Exercise 9C

Pearson Pure Mathematics Year 1/AS
Pages 184-185

Area of Non Right-Angled Triangles



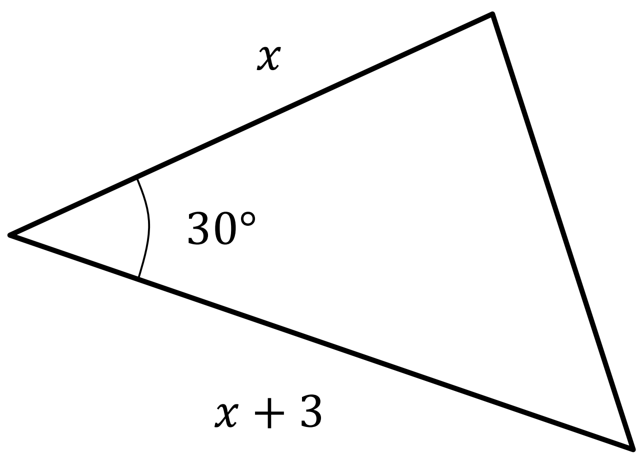
Area ?

$$\text{Area} = \frac{1}{2} a\, b \sin(C)$$

where C is the angle between two sides a and b.

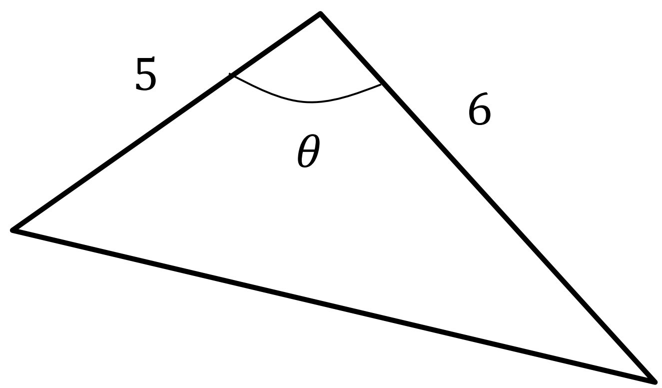
Fro Tip: You shouldn't have to label sides/angles before using the formula. Just remember that the angle is <u>between the two sides</u>.

Test Your Understanding



The area of this triangle is 10.
Determine x.

?



The area of this triangle is also 10.
If θ is obtuse, determine θ.

?

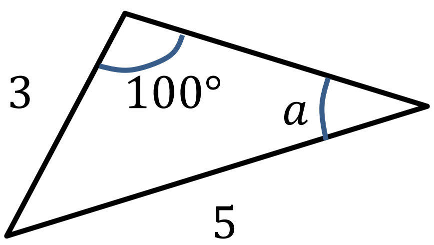
Exercise 9D

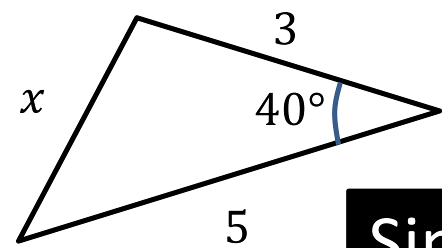
Pearson Pure Mathematics Year 1/AS
Pages 186-187

Sin or cosine rule?

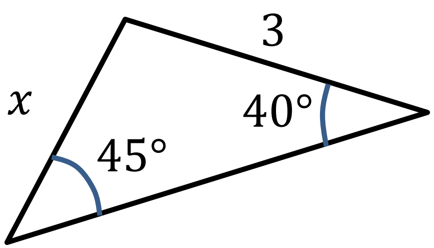
Sine | Cosine

Recall that whenever we have **two "side-angle pairs**" involved, use sine rule. If there's **3 sides** involved, we can use cosine rule. Sine rule is generally easier to use than cosine rule.

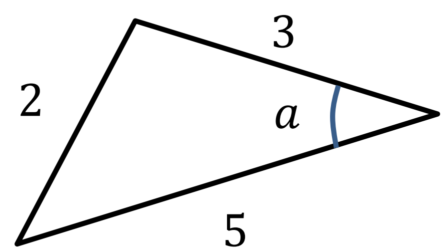




Sine | Cosine



Sine | Cosine

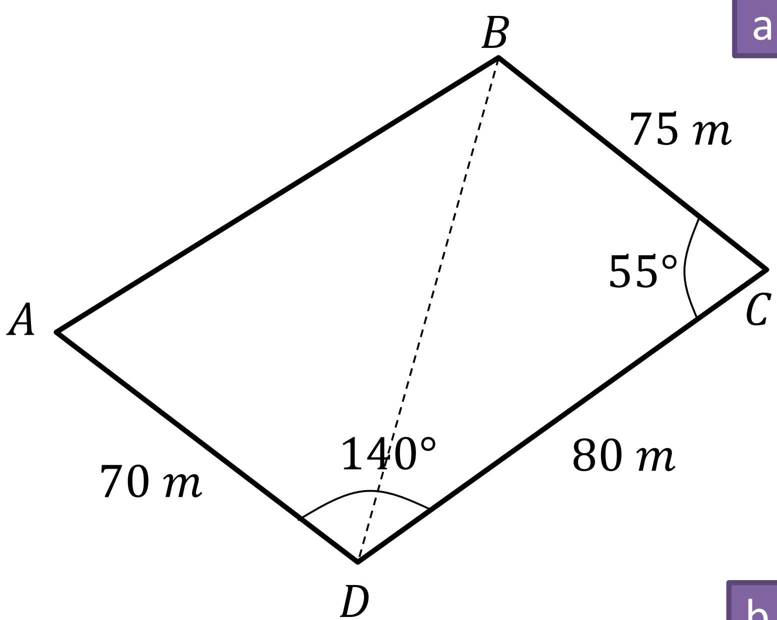


Sine | Cosine

Problem Solving With Sine/Cosine Rule

[From Textbook] The diagram shows the locations of four mobile phone masts in a field, $BC = 75\ m$. $CD = 80m$, angle $BCD = 55°$ and angle $ADC = 140°$.
In order that the masts do not interfere with each other, they must be at least 70m apart.
Given that A is the minimum distance from D, find:

a) The distance A is from B
b) The angle BAD
c) The area enclosed by the four masts.



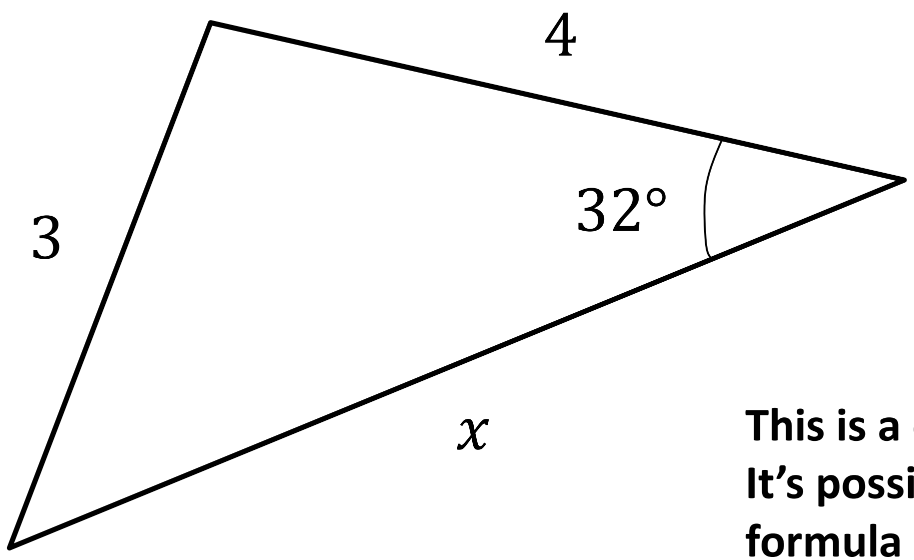
a ?

b ?

c ?

Using sine rule twice

You have	You want	Use
#4 Two sides known and a missing side <u>not</u> opposite known angle	Remaining side	Sine rule twice



Given there is just one angle involved, you might attempt to use the cosine rule:

?

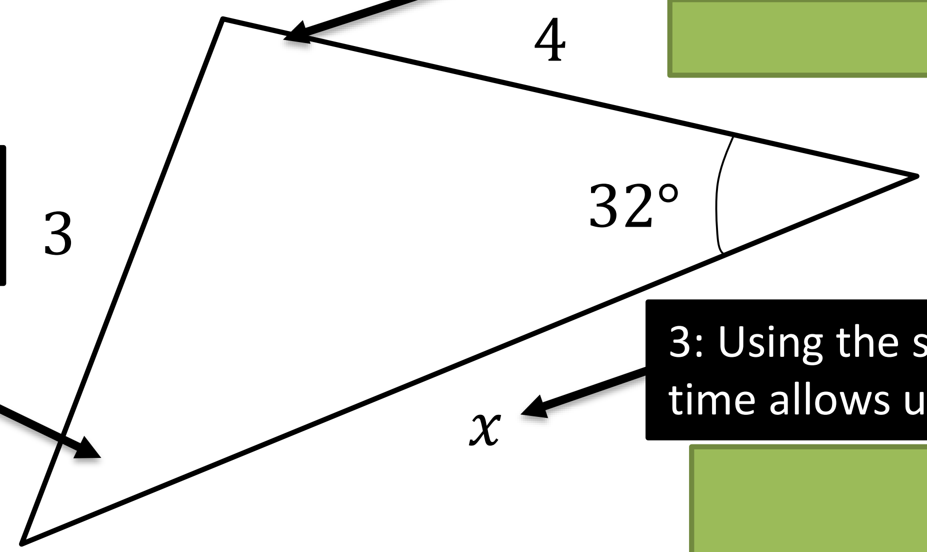
This is a quadratic equation!
It's possible to solve this using the quadratic formula (using $a = 1, b = -8\cos 32$, $c = 7$). However, this is a bit fiddly and not the primary method expected in the exam…

Using sine rule twice

You have	You want	Use
#4 Two sides known and a missing side <u>not</u> opposite known angle	Remaining side	Sine rule twice

2: Which means we would then know this angle.

?



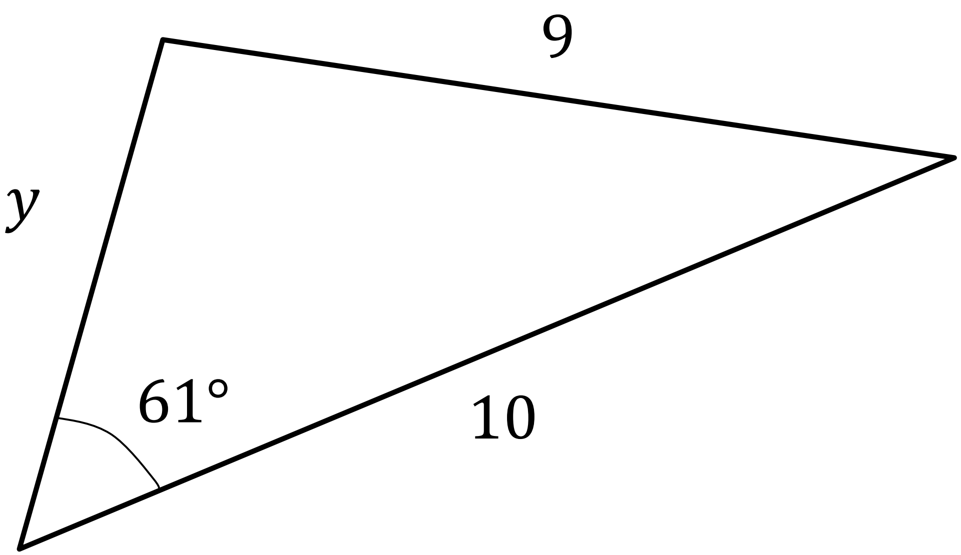
1: We could use the sine rule to find this angle.

?

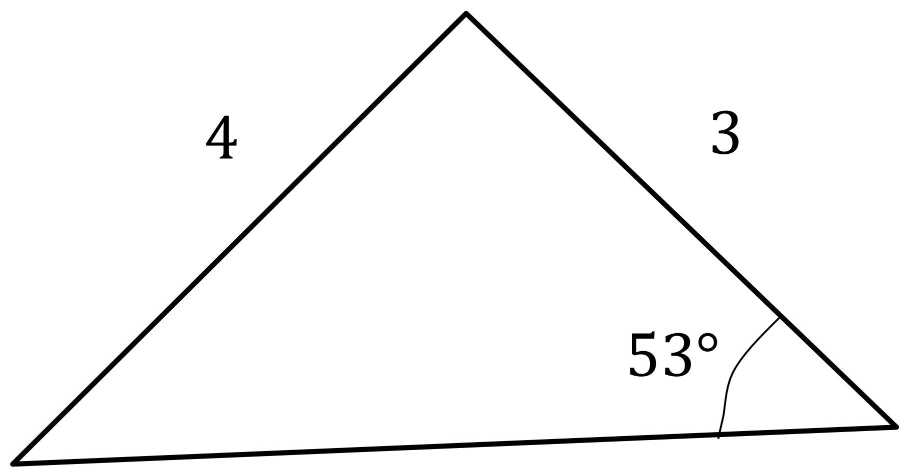
3: Using the sine rule a second time allows us to find x

?

Test Your Understanding



$y =$?



$Area =$?

Exercise 9E

Pearson Pure Mathematics Year 1/AS
Pages 189-191

1 [AEA 2009 Q5a] The sides of the triangle ABC have lengths $BC = a, AC = b$ and $AB = c$, where $a < b < c$. The sizes of the angles A, B and C form an arithmetic sequence.

(i) Show that the area of triangle ABC is $ac\frac{\sqrt{3}}{4}$.

Given that $a = 2$ and $\sin A = \frac{\sqrt{15}}{5}$, find

(ii) the value of b,

(iii) the value of c.

2 [STEP I 2006 Q8] *Note that the volume of a tetrahedron is equal to*

$$\frac{1}{3} \times area\ of\ base \times height$$

The points O, A, B, C have coordinates $(0,0,0), (a, 0,0), (0, b, 0)$ and $(0,0, c)$ respectively, where a, b, c are positive.

(i) Find, in terms of a, b, c the volume of the tetrahedron $OABC$.

(ii) Let angle $ACB = \theta$. Show that

$$\cos\theta = \frac{c^2}{\sqrt{(a^2+c^2)(b^2+c^2)}}$$

and find, in terms of a, b and c, the area of triangle ABC.

Hence show that d, the perpendicular distance of the origin from the triangle ABC, satisfies $\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

Solutions to extension problems on next slides.

Solution to Extension Problem 1

[AEA 2009 Q5a] The sides of the triangle ABC have lengths $BC = a, AC = b$ and $AB = c$, where $a < b < c$. The sizes of the angles A, B and C form an arithmetic sequence.

(i) Show that the area of triangle ABC is $ac\frac{\sqrt{3}}{4}$.

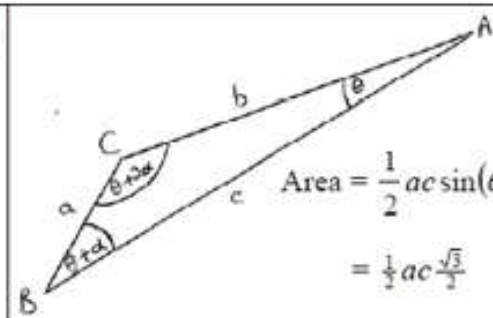
Given that $a = 2$ and $\sin A = \frac{\sqrt{15}}{5}$, find

(ii) the value of b,

(iii) the value of c.

(a)

(i)



$$\theta + (\theta+\alpha) + (\theta + 2\alpha) = 180$$

$$3\theta + 3\alpha = 180$$

$$\therefore \hat{B} = (\theta+\alpha) = 60^\circ$$

$$\text{Area} = \frac{1}{2}ac\sin(\theta+\alpha)$$

$$= \tfrac{1}{2}ac\tfrac{\sqrt{3}}{2} \qquad = \frac{ac\sqrt{3}}{4} \qquad (*)$$

(ii) Sine Rule

$$\frac{b}{\sin(\theta+\alpha)} = \frac{a}{\sin A} \quad \text{OR} \quad \frac{1}{2}\, bc\sin A = \frac{ac\sqrt{3}}{4}$$

$$\therefore \quad b = 2\times\frac{5}{\sqrt{15}}\times\frac{\sqrt{3}}{2} = \underline{\sqrt{5}}$$

(iii) Cosine Rule

$$b^2 = a^2 + c^2 - 2ac\cos(\theta+\alpha)$$

$$5 = 4 + c^2 - 2\times 2\times c\ \frac{1}{2}$$

$$0 = c^2 - 2c - 1 \quad \text{OR} \quad c^2 - 2\sqrt{2} + 1 = 0$$

$$c = \frac{2\pm\sqrt{4+4}}{2}$$

$$c = \underline{1+\sqrt{2}} \qquad \text{OR} \qquad (3+2\sqrt{2})^{1/2}$$

Solution to Extension Problem 2

[STEP I 2006 Q8] Note that the volume of a tetrahedron is equal to $\frac{1}{3} \times area\ of\ base \times height$

The points O, A, B, C have coordinates $(0,0,0), (a,0,0), (0,b,0)$ and $(0,0,c)$ respectively, where a, b, c are positive.

(i) Find, in terms of a, b, c the volume of the tetrahedron $OABC$.

(ii) Let angle $ACB = \theta$. Show that
$$\cos\theta = \frac{c^2}{\sqrt{(a^2+c^2)(b^2+c^2)}}$$
and find, in terms of a, b and c, the area of triangle ABC.
Hence show that d, the perpendicular distance of the origin from the triangle ABC, satisfies $\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

(i) The volume of $OABC = \frac{1}{3} \times$ the area of triangle $OAB \times OC = \frac{1}{6}abc$.

(ii) Using the scalar product with vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$,

This is FM content, but see a few lines below.

$$\sqrt{a^2+c^2}\sqrt{b^2+c^2} \times \cos\theta = \begin{pmatrix} a \\ 0 \\ -c \end{pmatrix} \cdot \begin{pmatrix} 0 \\ b \\ -c \end{pmatrix} = c^2 \Rightarrow \cos\theta = \frac{c^2}{\sqrt{a^2+c^2}\sqrt{b^2+c^2}}$$

The cosine rule ($AB^2 = AC^2 + BC^2 - 2 \times AC \times BC \times \cos\theta$) will also yield this result.

The area of triangle ABC will be $\frac{1}{2} \times \sqrt{a^2+c^2}\sqrt{b^2+c^2} \times \sin\theta$

$$= \frac{1}{2} \times \sqrt{a^2+c^2}\sqrt{b^2+c^2} \times \sqrt{1 - \left(\frac{c^2}{\sqrt{a^2+c^2}\sqrt{b^2+c^2}}\right)^2} \quad \text{(because } \sin^2\theta \equiv 1 - \cos^2\theta\text{)}$$

$$= \frac{1}{2} \times \sqrt{(a^2+c^2)(b^2+c^2) - c^4}$$

$$= \frac{1}{2} \times \sqrt{a^2b^2 + b^2c^2 + c^2a^2}$$

So $\frac{1}{3} \times \left(\frac{1}{2} \times \sqrt{a^2b^2+b^2c^2+c^2a^2}\right) \times d = \frac{1}{6}abc \Rightarrow \frac{1}{d^2} = \frac{a^2b^2+b^2c^2+c^2a^2}{a^2b^2c^2}$

which simplifies to the stated result.

A similar result is true for the right-angled triangle PQR, in which X is the foot of the perpendicular from the right-angle Q to the hypotenuse PR: $\frac{1}{PQ^2} + \frac{1}{QR^2} = \frac{1}{QX^2}$

Sin Graph

What does it look like?

?

Cos Graph

What do the following graphs look like?

?

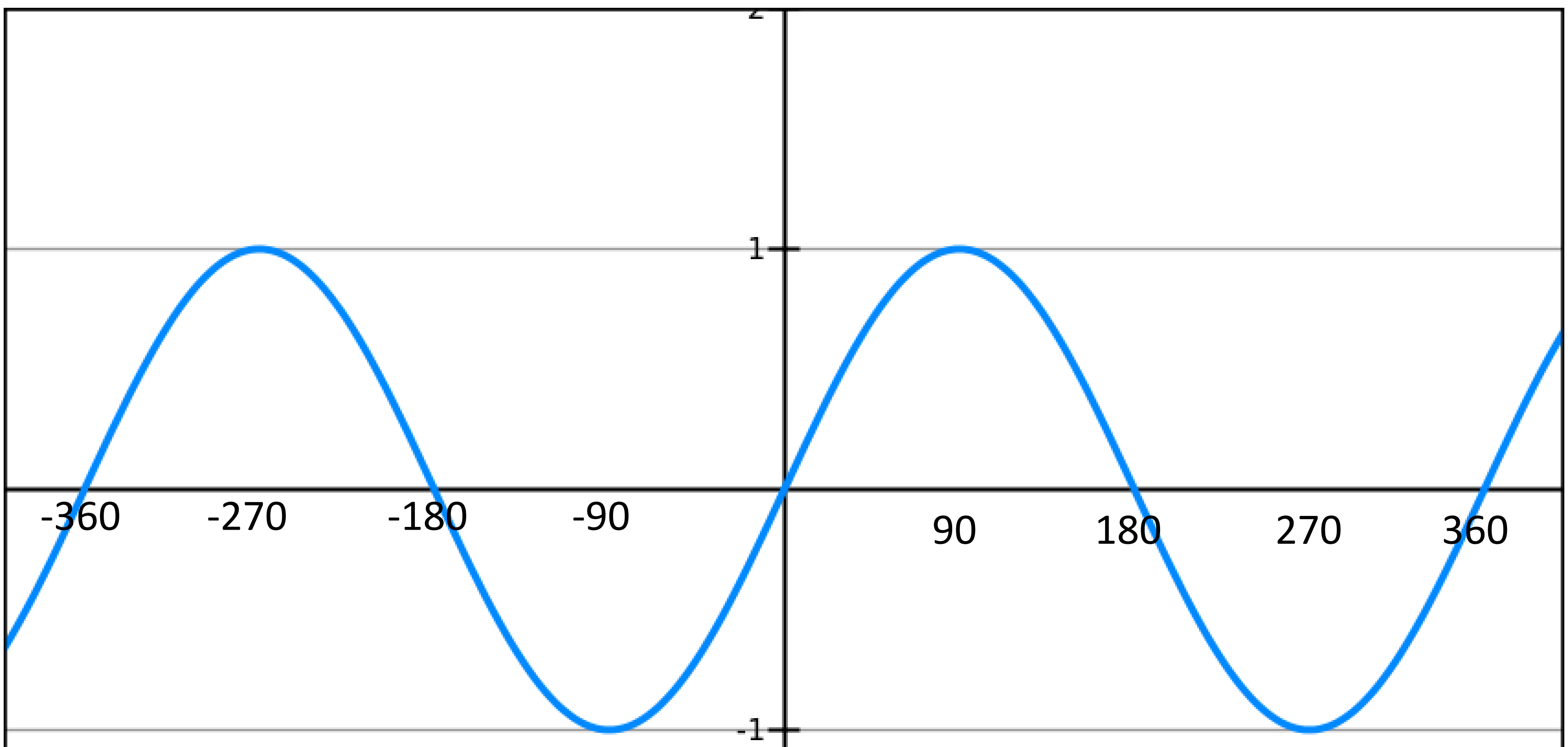
Tan Graph

What does it look like?

?

Sin Graph

What do the following graphs look like?



Suppose we know that **sin(30) = 0.5**. By thinking about symmetry in the graph, how could we work out:

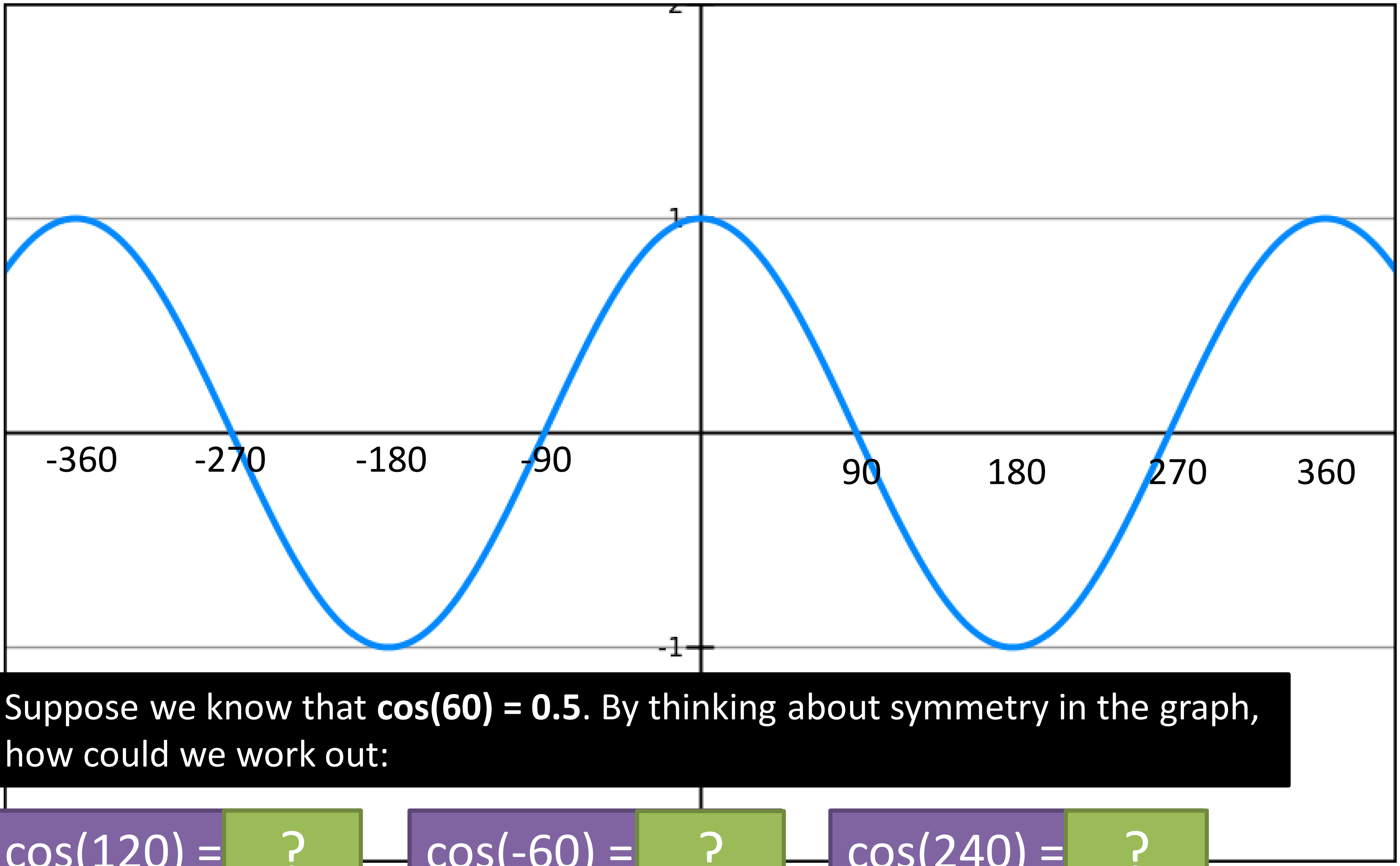
sin(150) = ?

sin(-30) = ?

sin(210) = ?

Cos Graph

What does it look like?



Suppose we know that **cos(60) = 0.5**. By thinking about symmetry in the graph, how could we work out:

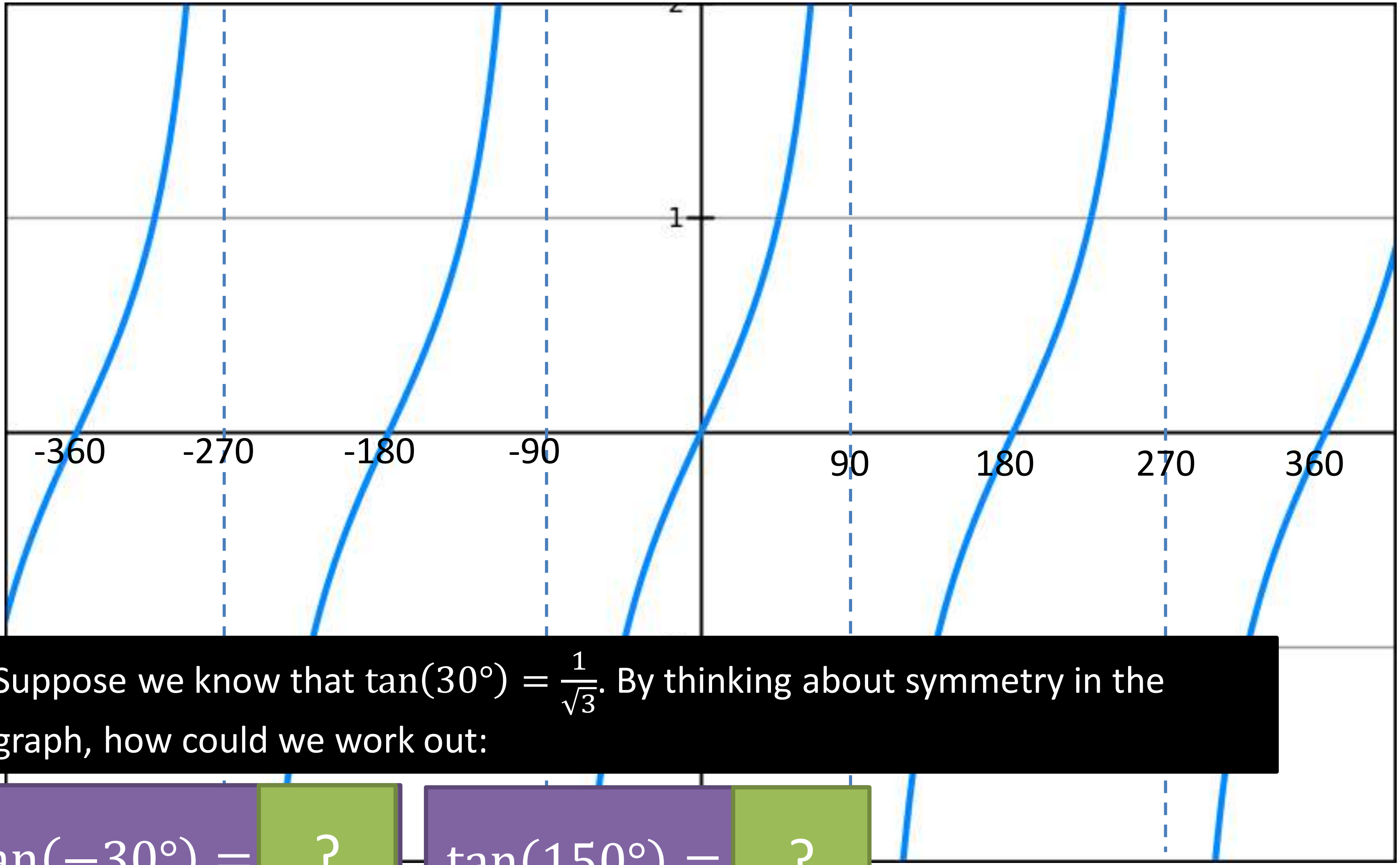
cos(120) = ?

cos(-60) = ?

cos(240) = ?

Tan Graph

What does it look like?



Suppose we know that $\tan(30°) = \frac{1}{\sqrt{3}}$. By thinking about symmetry in the graph, how could we work out:

$\tan(-30°) =$?

$\tan(150°) =$?

Transforming Trigonometric Graphs

There is no new theory here: just use your knowledge of transforming graphs, i.e. whether the transformation occurs 'inside' the function (i.e. input modified) or 'outside' the function (i.e. output modified).

Sketch $y = 4\sin x$, $0 \leq x \leq 360°$

?

Sketch $y = \cos(x + 45°)$, $0 \leq x \leq 360°$

?

Transforming Trigonometric Graphs

Sketch $y = -\tan x, 0 \leq x \leq 360°$

?

Sketch $y = \sin\left(\frac{x}{2}\right), 0 \leq x \leq 360°$

?

Exercise 9F/9G

Pearson Pure Mathematics Year 1/AS
Pages 194, 197-198

Extension

1 *[MAT 2013 1B]* The graph of $y = \sin x$ is reflected first in the line $x = \pi$ and then in the line $y = 2$. The resulting graph has equation:

A) $y = \cos x$
B) $y = 2 + \sin x$
C) $y = 4 + \sin x$
D) $y = 2 - \cos x$

?

2 *[MAT 2011 1D]* What fraction of the interval $0 \leq x \leq 360°$ is one (or both) of the inequalities:

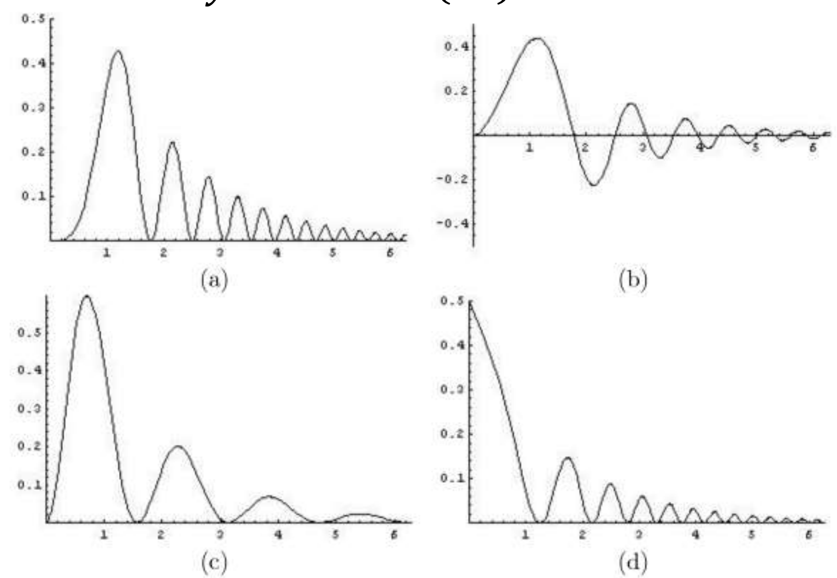
$$\sin x \geq \frac{1}{2}, \qquad \sin 2x \geq \frac{1}{2}$$

true?

?

3 *[MAT 2007 1G]* On which of the axes is a sketch of the graph

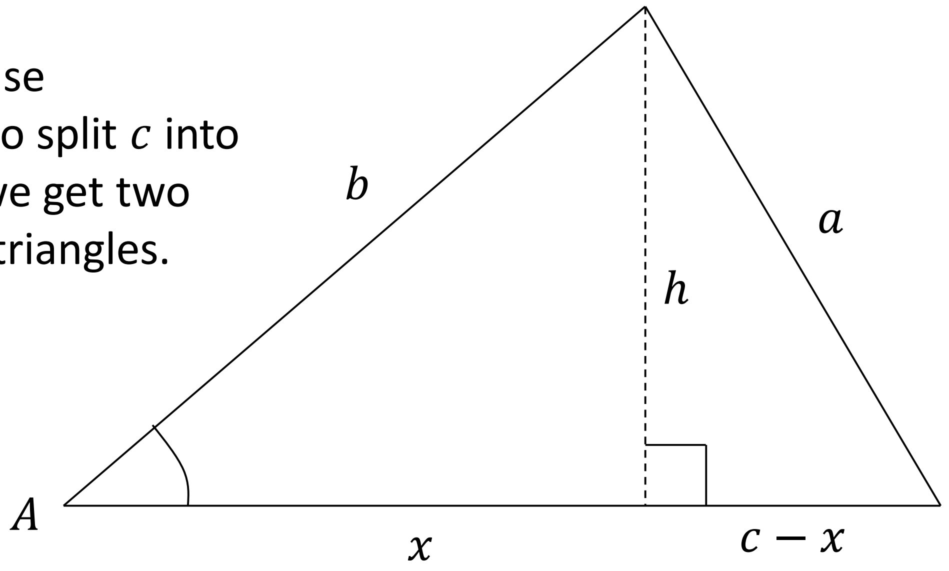
$$y = 2^{-x} \sin^2(x^2)$$



?

APPENDIX :: Proof of Cosine Rule

We want to use Pythagoras, so split c into two so that we get two right-angled triangles.



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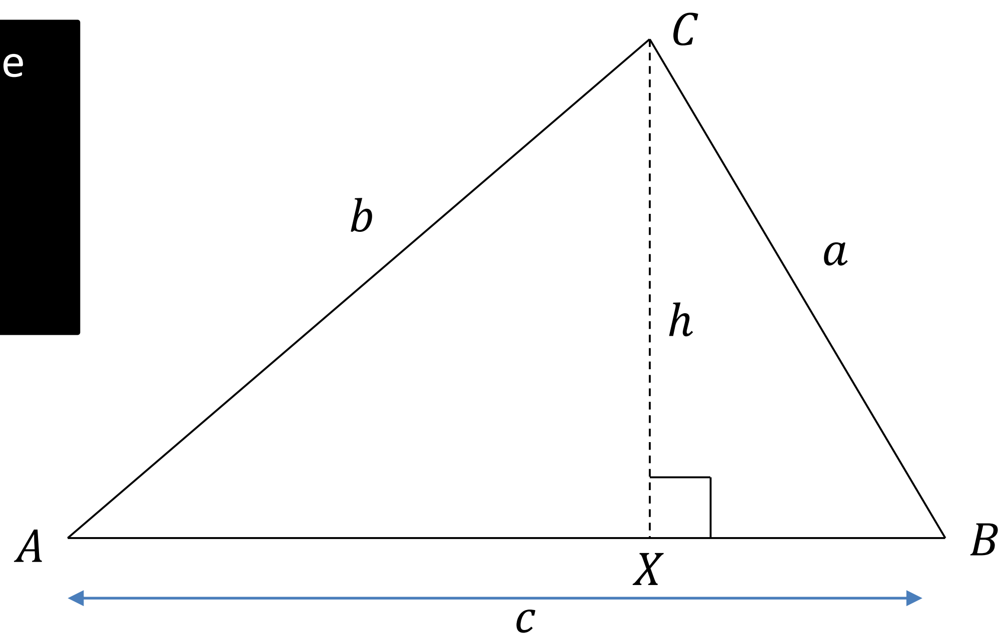
?

APPENDIX :: Proof of Sine Rule

The idea is that we can use the common length of ΔACX and $\angle XBC$, i.e. h, to connect the two triangles, and therefore connect their angles/length.



Using basic trigonometry:

?

?