P1 Chapter 11 :: Vectors

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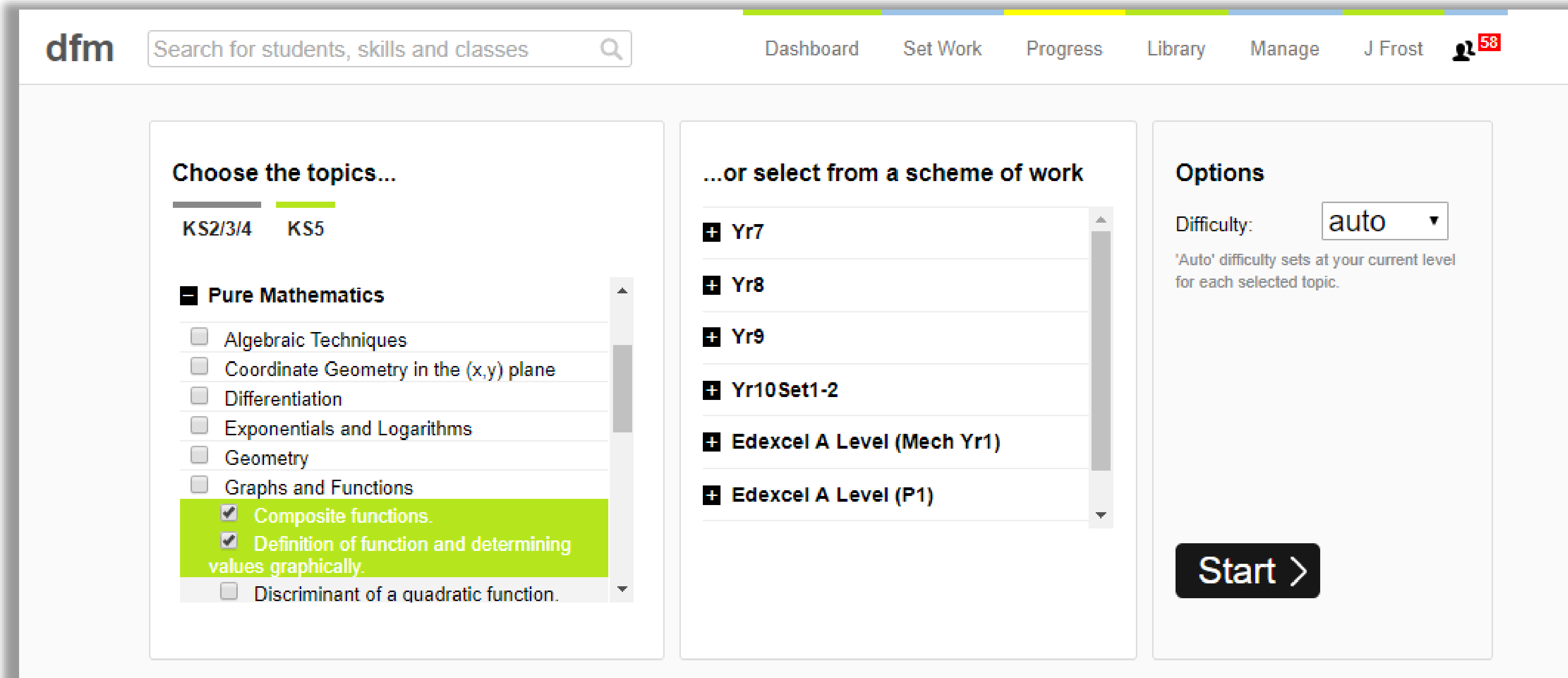

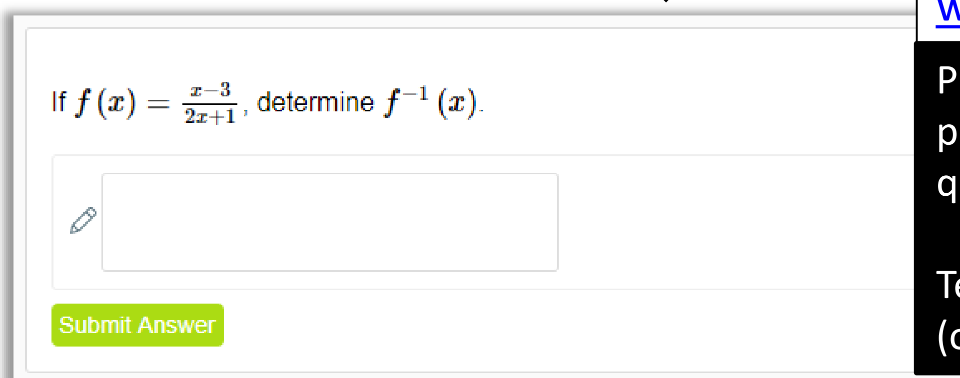

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Chapter Overview

Vectors used to be a Year 2 A Level topic. However, the more difficult content has been moved to Further Maths. This chapter is now more intended as providing the 'fundamentals' of vectors (most of which you learned at GCSE), supporting Mechanics and FM.

1:: Add/scale factors and show vectors are parallel.

2:: Calculate magnitude and direction of a vector.

If $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$, determine $|\boldsymbol{a}|$.

3:: Understand and use position vectors.

A donut has position vector $4\boldsymbol{i} + 3\boldsymbol{j}$...

4:: Solve both geometric problems.

An orienteer leaves position O and walks 15km at a bearing of $120°$ to the position A. Determine $\overrightarrow{OA}$.

5:: Understand speed vs velocity

If a ship moves at a velocity of $(12i + 5j)\ km/h$, what is its speed?

Vector Basics

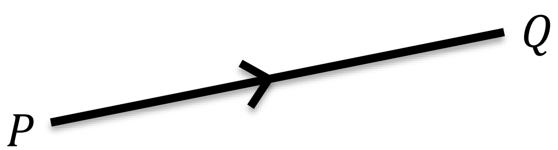
A

Whereas a **coordinate** represents a **position** in space, a **vector** represents a **displacement** in space.

A vector has 2 properties:

- Direction
- Magnitude (i.e. length)

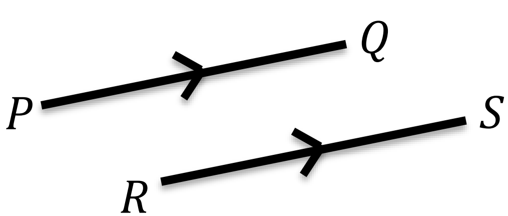
If P and Q are points then $\overrightarrow{PQ}$ is the vector between them.



B

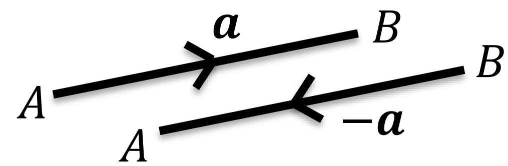
If two vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same magnitude and direction, **they're the same vector** and are **parallel**.

This might seem obvious, but students sometimes think the vector is different because the movement occurred at a different point in space. Nope!



C

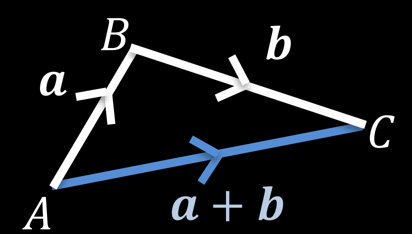
$\overrightarrow{AB} = -\overrightarrow{BA}$ and the two vectors are parallel, equal in magnitude but in **opposite directions**.



D

Triangle Law for vector addition:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

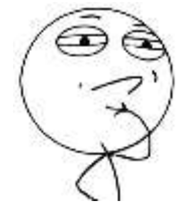


The vector of multiple vectors is known as the **resultant vector.**

(you will encounter this term in Mechanics)

Just for your interest…

Have you ever wondered what happens if you 'multiply' two vectors or two sets?



Erm…

In KS2/3 you probably only experienced variables holding numerical values. You since saw that variables can represent other mathematical types, such as sets or vectors:

$$x = 3$$
$$\boldsymbol{a} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$
$$T = \{2,5,9,10\}$$

But when we do we need to define **explicitly** what operators like '+' and '×' mean.

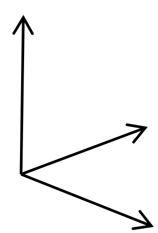
$$1 + 3 = 4 \qquad \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = ? \qquad \{a, b, c\} \times \{d, e\} = ?$$

Often these operators are defined to give it properties that are consistent with its usage elsewhere, e.g. 'commutativity': $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ for vector addition just as $2 + 4 = 4 + 2$ for numbers.

In FM, you will see that 'multiplying' two 3D vectors (known as the **cross product**) gives you a vector **perpendicular to the two**.
Vector multiplication is not commutative, so $\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{b} \times \boldsymbol{a}$ (however it is 'distributive', so $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{c})$)
The Casio Classwiz can calculate this in Vector mode!

$$\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$$



'Multiplying' **two sets** (known again as the cross product) finds each possible combination of members, one from each:

$$\{a, b, c\} \times \{d, e\} = \{\{a, d\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, d\}, \{c, e\}\}$$

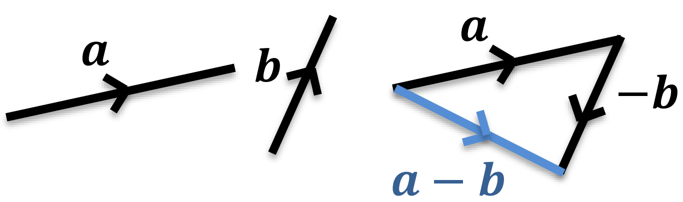
It has the nice property that $n(A \times B) = n(A) \times n(B)$, where $n(A)$ gives the size of the set A. Also, $A \times B = B \times A$.

ME-WOW!

Vector Basics

E Vector **subtraction** is defined using vector addition and negation:

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$$



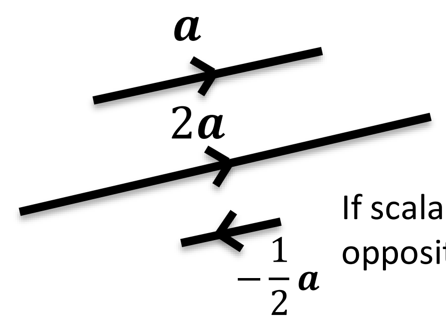
F The zero vector $\mathbf{0}$ (a bold 0), represents no movement.

$$\overrightarrow{PQ} + \overrightarrow{QP} = \mathbf{0}$$

In 2D: $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

G A **scalar** is a normal number, which can be used to 'scale' a vector.

- The **direction** will be the **same**.
- But the **magnitude** will be **different** (unless the scalar is 1).



If scalar is negative, opposite direction.

H Any vector parallel to the vector $\boldsymbol{a}$ can be written as $\lambda\boldsymbol{a}$, where λ is a scalar.

The implication is that if we can write one vector **as a multiple of** another, then we can show they are parallel.

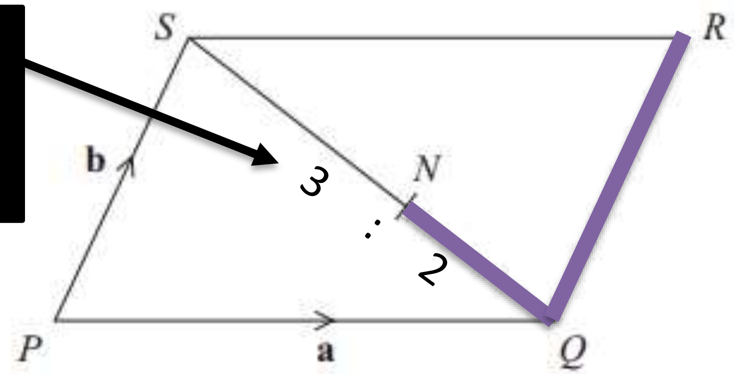
"Show $2\boldsymbol{a} + 4\boldsymbol{b}$ and $3\boldsymbol{a} + 6\boldsymbol{b}$ are parallel".

$3\boldsymbol{a} + 6\boldsymbol{b} = \frac{3}{2}(\boldsymbol{a} + 2\boldsymbol{b})$ ∴ parallel

Example

Edexcel GCSE June 2013 1H Q27

Fro Tip: This ratio wasn't in the original diagram. I like to add the ratio as a visual aid.



$PQRS$ is a parallelogram.
N is the point on SQ such that $SN : NQ = 3 : 2$

$\overrightarrow{PQ} = \mathbf{a}$ $\quad \overrightarrow{PS} = \mathbf{b}$

(a) Write down, in terms of **a** and **b**, an expression for $\overrightarrow{SQ}$.

(b) Express $\overrightarrow{NR}$ in terms of **a** and **b**.

a $\quad \overrightarrow{SQ} =$?

For (b), there's two possible paths to get from N to R: via S or via Q. But which is best?

?

b ?

Fro Workings Tip: While you're welcome to start your working with the second line, I recommend the first line so that your chosen route is clearer.

Test Your Understanding

Edexcel GCSE June 2012

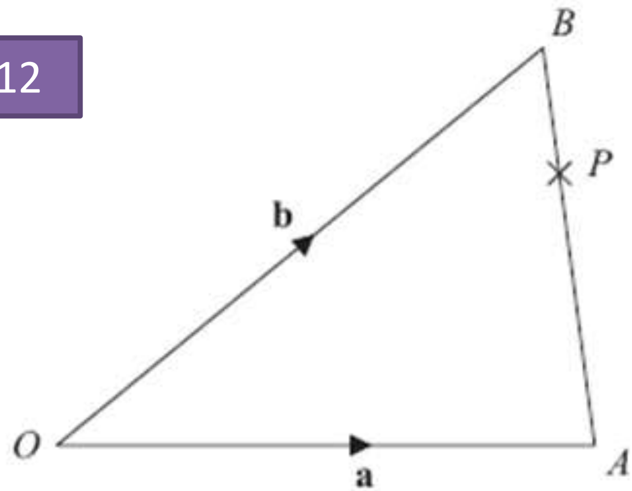


Diagram **NOT** accurately drawn

OAB is a triangle.

$\overrightarrow{OA} = \mathbf{a}$
$\overrightarrow{OB} = \mathbf{b}$

(a) Find $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

?

....

(1)

P is the point on AB such that $AP : PB = 3 : 1$

(b) Find $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
Give your answer in its simplest form.

?

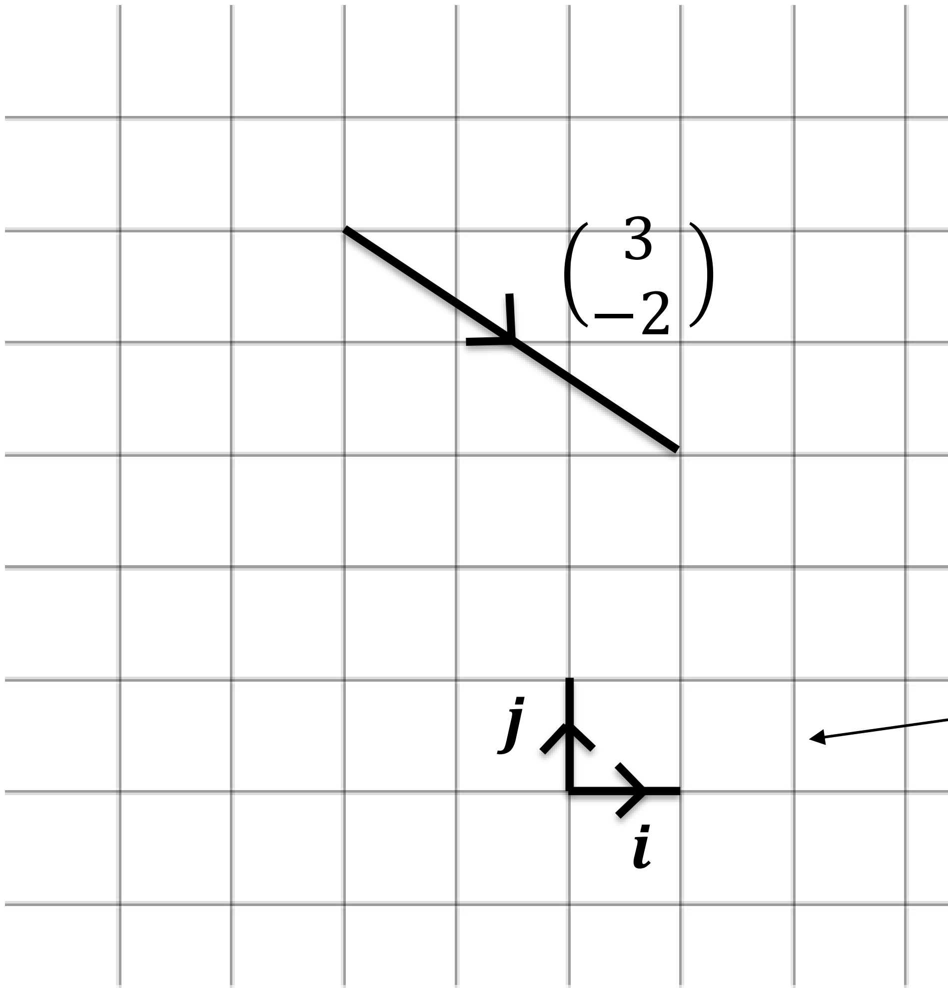
....

Exercise 11A

Pearson Pure Mathematics Year 1/AS

Pages 234-235

Representing Vectors



You should already be familiar that the value of a vector is the **displacement** in the x and y direction (if in 2D).

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$\boldsymbol{a} + \boldsymbol{b} =$?

$2\boldsymbol{a} =$?

✏ A **unit vector** is a vector of magnitude 1. $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the x-axis and y-axis respectively.

$$\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

e.g. $\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 4\boldsymbol{i} + 3\boldsymbol{j}$

Note: This allows us to write any vector algebraically without using vector notation. **Any point in 2D space,** as a vector from the origin, can be obtained using a linear combination of $\boldsymbol{i}$ and $\boldsymbol{j}$, e.g. if $P(5,-1)$, $\overrightarrow{OP} = 5\boldsymbol{i} - \boldsymbol{j}$. For this reason, $\boldsymbol{i}$ and $\boldsymbol{j}$ are known as **basis vectors** of 2D coordinate space. In fact, any two non-parallel/non-zero vectors can be used as basis vectors, e.g. if $\boldsymbol{a} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, it's possible to get any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ using a linear combination of these, i.e. we can always find scalars p and q such that $\begin{pmatrix} x \\ y \end{pmatrix} = p\begin{pmatrix} 5 \\ 2 \end{pmatrix} + q\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

Examples

If $\boldsymbol{a} = 3\boldsymbol{i}, \quad \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}, \quad \boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$ then:

1) Write $\boldsymbol{a}$ in vector form.
2) Find $\boldsymbol{b} + 2\boldsymbol{c}$ in $\boldsymbol{i}, \boldsymbol{j}$ form.

1 ?

2 ?

4 Given that $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j}$, find:

a λ if $\mathbf{a} + \lambda\mathbf{b}$ is parallel to the vector $\mathbf{i}$

b μ if $\mu\mathbf{a} + \mathbf{b}$ is parallel to the vector $\mathbf{j}$

5 Given that $\mathbf{c} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{d} = \mathbf{i} - 2\mathbf{j}$, find:

a λ if $\mathbf{c} + \lambda\mathbf{d}$ is parallel to $\mathbf{i} + \mathbf{j}$

b μ if $\mu\mathbf{c} + \mathbf{d}$ is parallel to $\mathbf{i} + 3\mathbf{j}$

c s if $\mathbf{c} - s\mathbf{d}$ is parallel to $2\mathbf{i} + \mathbf{j}$

d t if $\mathbf{d} - t\mathbf{c}$ is parallel to $-2\mathbf{i} + 3\mathbf{j}$

Exercise 11B

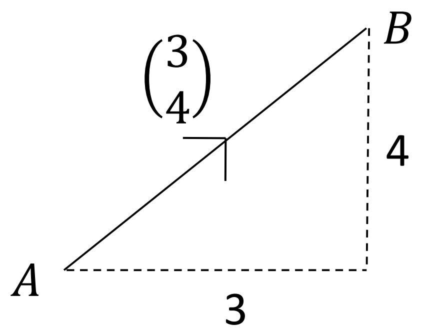
Pearson Pure Mathematics Year 1/AS
Pages 237-238

Magnitude of a Vector

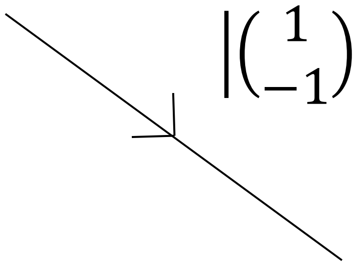
The magnitude $|a|$ of a vector a is its length.

If $\boldsymbol{a} = \begin{pmatrix} x \\ y \end{pmatrix} = xi + yj$

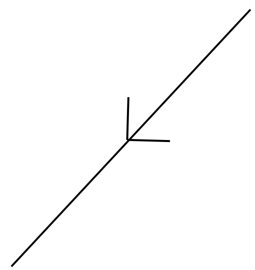
$|\boldsymbol{a}| = \sqrt{x^2 + y^2}$



$\left|\overrightarrow{AB}\right| =$?



$\left|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right| =$?



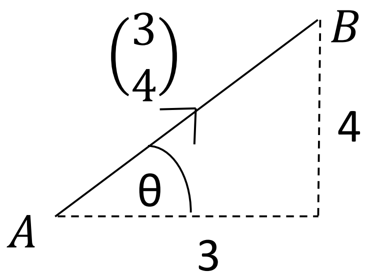
$\left|\begin{pmatrix} -5 \\ -12 \end{pmatrix}\right| =$?

$\boldsymbol{a} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ $\quad |\boldsymbol{a}| =$?

$\boldsymbol{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\quad |\boldsymbol{b}| =$?

Direction of a Vector

The direction of a vector can be found using basic trigonometry.



Magnitude

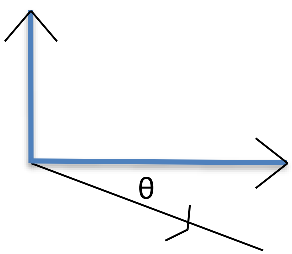
$$\left|\overrightarrow{AB}\right| = \sqrt{3^2 + 4^2} = 5$$

Angle with ***i*** **:**

?

Examples:

1. Find the angle that vector $\boldsymbol{a} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ makes with the positive x axis.

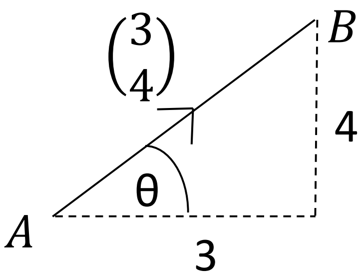


?

Angle with x axis =

Direction of a Vector

✎ The direction of a vector can be found using basic trigonometry.



Magnitude

$$|\overrightarrow{AB}| = \sqrt{3^2+4^2} = 5$$

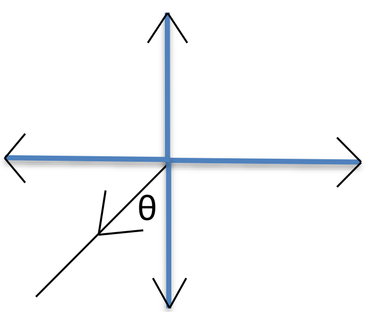
Angle with ***i*** **:**

$$\tan\theta = \frac{4}{5}$$

$$\theta = 38.7\,°$$

Examples:

2. Find the angle that vector $\boldsymbol{b} = \begin{pmatrix}-5\\-12\end{pmatrix}$ makes with ***j***.



$Angle$ with x axis

?

Unit Vectors

A unit vector is a vector whose magnitude is 1

There's certain operations on vectors that require the vectors to be 'unit' vectors. We just scale the vector so that its magnitude is now 1.

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

?

If $\boldsymbol{a}$ is a vector, then the unit vector $\hat{\boldsymbol{a}}$ in the same direction is

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

Test Your Understanding: Convert the following vectors to unit vectors.

$$\boldsymbol{a} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$$

?

$$\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

?

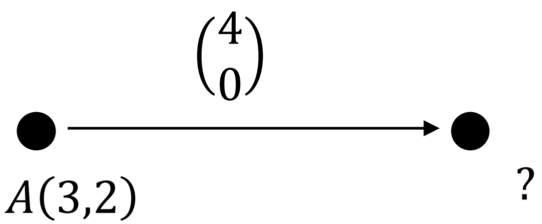
Exercise 11C

Pearson Pure Mathematics Year 1/AS

Pages 240-242

Position Vectors

Suppose we started at a point $(3,2)$ and translated by the vector $\begin{pmatrix}4\\0\end{pmatrix}$:

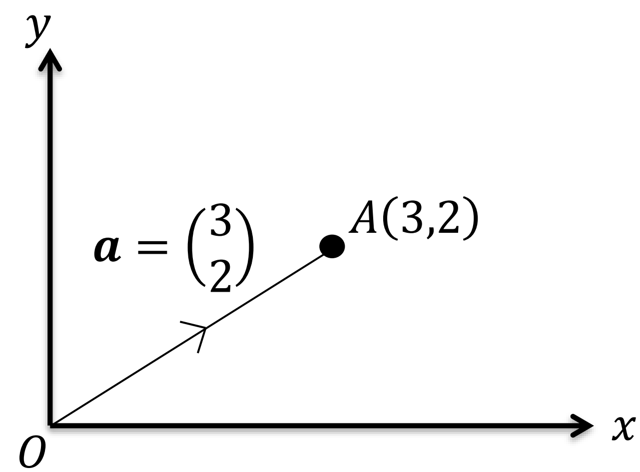


You might think we can do something like:

$$(3,2) + \begin{pmatrix}4\\0\end{pmatrix} = (7,2)$$

But only vectors can be added to other vectors. **If we treated the point $(3,2)$ as a vector**, then this solves the problem:

$$\begin{pmatrix}3\\2\end{pmatrix} + \begin{pmatrix}4\\0\end{pmatrix} = \begin{pmatrix}7\\2\end{pmatrix}$$



A vector used to represent a position is unsurprisingly known as a **position vector**.
A position can be thought of as a translation from the origin, as per above. It enables us to use positions in all sorts of vector (and matrix!) calculations.

The position vector of a point A is the vector $\overrightarrow{OA}$, where O is the origin. $\overrightarrow{OA}$ is usually written as $\boldsymbol{a}$.

Example

The points A and B have coordinates $(3,4)$ and $(11,2)$ respectively.
Find, in terms of i and j:

a) The position vector of A
b) The position vector of B
c) The vector $\overrightarrow{AB}$

a $\overrightarrow{OA} =$?

b $\overrightarrow{OB} =$?

c $\overrightarrow{AB} =$?

You can see this by inspection of the change in x and the change in y:

$+8$ -2

$A(3,4) \quad \rightarrow \quad B(11,2)$

More formally:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \binom{11}{2} - \binom{3}{4} = \binom{8}{-2}$$

Important!

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \boldsymbol{b} - \boldsymbol{a}$$

Further Example

$\overrightarrow{OA} = 5i - 2j$ and $\overrightarrow{AB} = 3i + 4j$. Find:

a) The position vector of B.

b) The exact value of $|\overrightarrow{OB}|$ in simplified surd form.

a $\overrightarrow{OB} =$?

b $|\overrightarrow{OB}| =$?

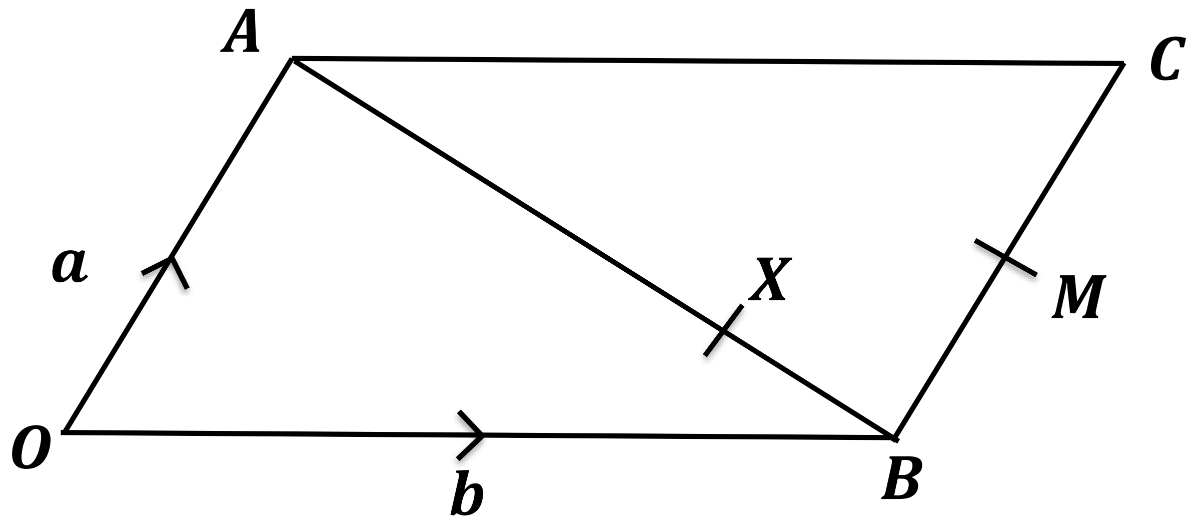
Either a quick sketch will help you see this, or thinking of $\overrightarrow{OA}$ as the original position and $\overrightarrow{AB}$ as the translation.

Exercise 11D

Pearson Pure Mathematics Year 1/AS

Pages 243-244

Solving Geometric Problems

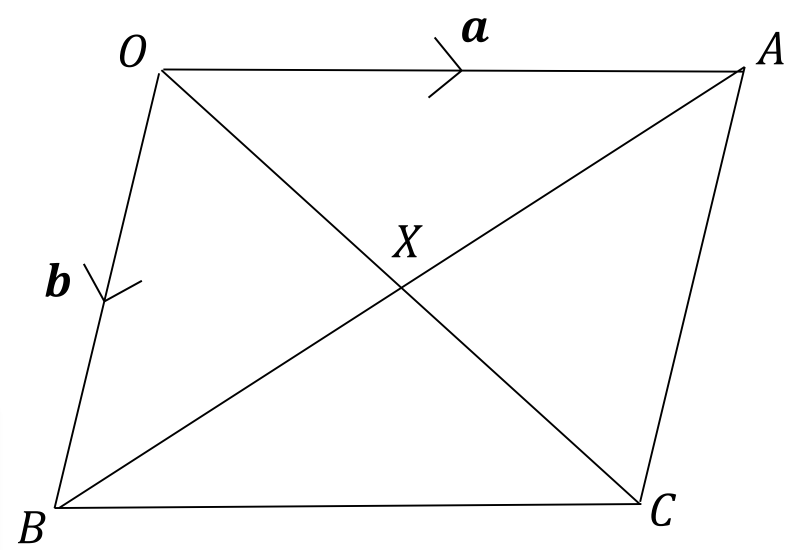


X is a point on AB such that $AX: XB = 3: 1$. M is the midpoint of BC.
Show that $\overrightarrow{XM}$ is parallel to $\overrightarrow{OC}$.

?

Introducing Scalars and Comparing Coefficients

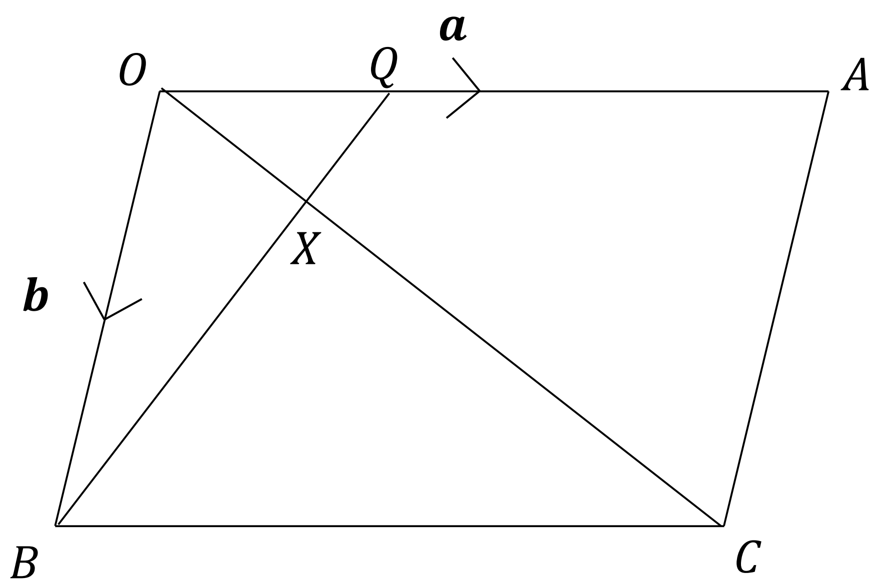
Remember when we had **identities** like: $ax^2 + 3x \equiv 2x^2 + bx$ we could **compare coefficients**, so that $a = 2$ and $3 = b$.
We can do the same with (non-parallel) vectors!



$OACB$ is a parallelogram, where $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. The diagonals OC and AB intersect at a point X. Prove that the diagonals bisect each other.
(Hint: Perhaps find $\overrightarrow{OX}$ in two different ways?)

?

Test Your Understanding



In the above diagram, $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OQ} = \frac{1}{3}\boldsymbol{a}$. We wish to find the ratio $OX : XC$.

a) If $\overrightarrow{OX} = \lambda\, \overrightarrow{OC}$, find an expression for $\overrightarrow{OX}$ in terms of $\boldsymbol{a}, \boldsymbol{b}$ and λ.

b) If $\overrightarrow{BX} = \mu\, \overrightarrow{BQ}$, find an expression for $\overrightarrow{OX}$ in terms of $\boldsymbol{a}, \boldsymbol{b}$ and μ.

c) By comparing coefficients or otherwise, determine the value of λ, and hence the ratio $OX : XC$.

a ?

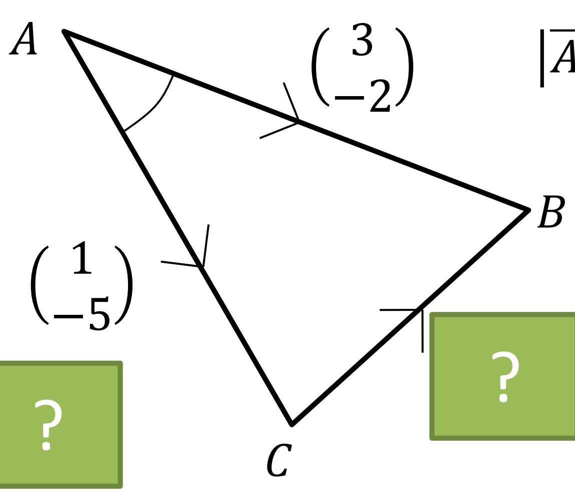
b ?

c ?

Area of a Triangle

$\overrightarrow{AB} = 3\boldsymbol{i} - 2\boldsymbol{j}$ and $\overrightarrow{AC} = \boldsymbol{i} - 5\boldsymbol{j}$. Determine $\angle BAC$.

Strategy: ?



$|\overrightarrow{AB}| =$?

$|\overrightarrow{CB}| =$?

$|\overrightarrow{AC}| =$?

?

Exercise 11E

Pearson Pure Mathematics Year 1/AS
Pages 246-247

Extension

1 *[STEP 2010 Q7]*
Relative to a fixed origin O, the points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. (The points O, A and B are not collinear.) The point C has position vector $\boldsymbol{c}$ given by

$$\boldsymbol{c} = \alpha\boldsymbol{a} + \beta\boldsymbol{b},$$

where α and β are positive constants with $\alpha + \beta < 1$. The lines OA and BC meet at the point P with position vector $\boldsymbol{p}$ and the lines OB and AC meet at the point Q with position vector $\boldsymbol{q}$. Show that

$$\boldsymbol{p} = \frac{\alpha a}{1-\beta}$$

and write down q in terms of α, β and b.
Show further that the point R with position vector r given by

$$\boldsymbol{r} = \frac{\alpha\boldsymbol{a} + \beta\boldsymbol{b}}{\alpha + \beta},$$

lies on the lines OC and AB.
The lines OB and PR intersect at the point S. Prove that $\frac{OQ}{BQ} = \frac{OS}{BS}$.

Click here for the solution:
http://www.mathshelper.co.uk/STEP%202010%20Solutions.pdf
(go to Q7)

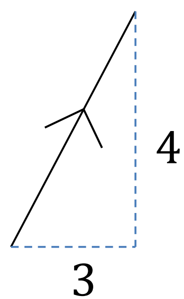
Modelling

In Mechanics, you will see certain things can be represented as a simple number (without direction), or as a vector (with direction):

Remember a 'scalar' just means a normal number (in the context of vectors). It can be obtained using the **magnitude** of the vector.

Vector Quantity	**Equivalent Scalar Quantity**
Velocity e.g. $\begin{pmatrix} 3 \\ 4 \end{pmatrix}\ km/h$	?
Displacement e.g. $\begin{pmatrix} -5 \\ 12 \end{pmatrix}\ km$	?

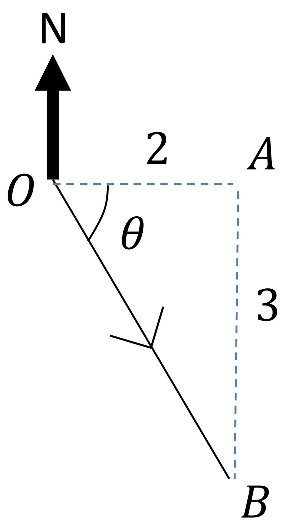
This means the position vector of the object changes by $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ each hour.



Example

[Textbook] A girl walks 2 km due east from a fixed point O to A, and then 3 km due south from A to B. Find

a) the total distance travelled
b) the position vector of B relative to O
c) $|\overrightarrow{OB}|$
d) The bearing of B from O.



a ?

b ?

c ?

d ?

Further Example

[Textbook] In an orienteering exercise, a cadet leaves the starting point O and walks 15 km on a bearing of $120°$ to reach A, the first checkpoint. From A he walks 9 km on a bearing of $240°$ to the second checkpoint, at B. From B he returns directly to O.

Find:

a) the position vector of A relative to O

b) $|\overrightarrow{OB}|$

c) the bearing of B from O

d) the position vector of B relative O.

I have no specific advice to offer except:

1. Draw a BIG diagram.
2. Remember bearings are measured clockwise from North.
3. Don't forget units (on vectors!)

?

b

?

c

?

Further Example

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?

Exercise 11F

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