U6 Chapter 3

Sequences and Series

Chapter Overview

1. Sequences
2. Arithmetic Series
3. Geometric Series
4. Sigma Notation
5. Recurrence Relations
6. Combined Sequences
7. Classifying Sequences

4 **Sequences and series** *continued*	4.2	Work with sequences including those given by a formula for the nth term and those generated by a simple relation of the form $x_{n+1} = \mathrm{f}(x_n)$; increasing sequences; decreasing sequences; periodic sequences.	For example $u_n = \frac{1}{3n+1}$ describes a decreasing sequence as $u_{n+1} < u_n$ for all integer n $u_n = 2^n$ is an increasing sequence as $u_{n+1} > u_n$ for all integer n $u_{n+1} = \frac{1}{u_n}$ for $n > 1$ and $u_1 = 3$ describes a periodic sequence of order 2
	4.3	Understand and use sigma notation for sums of series.	Knowledge that $\sum_1^n 1 = n$ is expected
	4.4	Understand and work with arithmetic sequences and series, including the formulae for nth term and the sum to n terms	The proof of the sum formula for an arithmetic sequence should be known including the formula for the sum of the first n natural numbers.
	4.5	Understand and work with geometric sequences and series, including the formulae for the nth term and the sum of a finite geometric series; the sum to infinity of a convergent geometric series, including the use of $\|r\| < 1$; modulus notation	The proof of the sum formula should be known. Given the sum of a series students should be able to use logs to find the value of n. The sum to infinity may be expressed as S_∞
	4.6	Use sequences and series in modelling.	Examples could include amounts paid into saving schemes, increasing by the same amount (arithmetic) or by the same percentage (geometric) or could include other series defined by a formula or a relation.

Sequences

A sequence is an ordered set of mathematical objects. Each element in the sequence is called a term.

$u_n =$

$n =$

Arithmetic Sequences

Examples

1. The nth term of an arithmetic sequence is $u_n = 55 - 2n$.

   a) Write down the first 3 terms of the sequence.

   b) Find the first term in the sequence that is negative.

2. Find the nth term of each arithmetic sequence.

a) 6, 20, 34, 48, 62

b) 101, 94, 87, 80, 73

3. A sequence is generated by the formula $u_n = an + b$ where a and b are constants to be found. Given that $u_3 = 5$ and $u_8 = 20$, find the values of the constants a and b.

4. For which values of x would the expression $-8,\ x^2$ and $17x$ form the first three terms of an arithmetic sequence.

Test Your Understanding

Xin has been given a 14 day training schedule by her coach.

Xin will run for A minutes on day 1, where A is a constant.

She will then increase her running time by $(d + 1)$ minutes each day, where d is a constant.

(a) Show that on day 14, Xin will run for

$$(A + 13d + 13) \text{ minutes.}$$

(2)

Yi has also been given a 14 day training schedule by her coach.

Yi will run for $(A - 13)$ minutes on day 1.

She will then increase her running time by $(2d - 1)$ minutes each day.

Given that Yi and Xin will run for the same length of time on day 14,

(b) find the value of d.

(3)

Extension

[STEP I 2004 Q5] The positive integers can be split into five distinct arithmetic progressions, as shown:

A: 1, 6, 11, 16, …
B: 2, 7, 12, 17, …
C: 3, 8, 13, 18, …
D: 4, 9, 14, 19, …
E: 5, 10, 15, 20, …

Write down an expression for the value of the general term in each of the five progressions.
Hence prove that the sum of any term in B and any term in C is a term in E.
Prove also that the square of every term in B is a term in D. State and prove a similar claim about the square of every term in C.

i) Prove that there are no positive integers x and y such that $x^2 + 5y = 243723$
ii) Prove also that there are no positive integers x and y such $x^4 + 2y^4 = 26081974$

Arithmetic Series

Proof of summation (required for exam):

Examples

1. Find the sum of the first 30 terms of the following arithmetic sequences

$2 + 5 + 8 + 11 + 14 \ldots$

$100 + 98 + 96 + \cdots$

$p + 2p + 3p + \cdots$

2. Find the greatest number of terms for the sum of $4 + 9 + 14 + \cdots$ to exceed 2000

Test Your Understanding

9. A company offers two salary schemes for a 10-year period, Year 1 to Year 10 inclusive.

Scheme 1: Salary in Year 1 is £P.
Salary increases by £$(2T)$ each year, forming an arithmetic sequence.

Scheme 2: Salary in Year 1 is £$(P + 1800)$.
Salary increases by £T each year, forming an arithmetic sequence.

(*a*) Show that the total earned under Salary Scheme 1 for the 10-year period is

$$£(10P + 90T).$$

(2)

For the 10-year period, the total earned is the same for both salary schemes.

(*b*) Find the value of T.

(4)

For this value of T, the salary in Year 10 under Salary Scheme 2 is £29 850.

(*c*) Find the value of P.

(3)

Extension

[MAT 2007 1J]
The inequality
$$(n+1)+(n^4+2)+(n^9+3)+\cdots \qquad +(n^{10000}+100) > k$$
Is true for all $n \geq 1$. It follows that
A) $k < 1300$
B) $k^2 < 101$
C) $k \geq 101^{10000}$
D) $k < 5150$

[AEA 2010 Q2]
The sum of the first p terms of an arithmetic series is q and the sum of the first q terms of the same arithmetic series is p, where p and q are positive integers and $p \neq q$.
Giving simplified answers in terms of p and q, find
a) The common difference of the terms in this series,
b) The first term of the series,
c) The sum of the first $(p+q)$ terms of the series.

[MAT 2008 1I]
The function $S(n)$ is defined for positive integers n by
$S(n) =$ sum of digits of n
For example, $S(723) = 7 + 2 + 3 = 12$.
The sum
$S(1) + S(2) + S(3) + \cdots + S(99)$
equals what?

Ex 3B Pg 64

Geometric Series

Identify the common ratio r:

1. $1, 2, 4, 8, 16, 32, \ldots$
2. $27, 18, 12, 8, \ldots$
3. $10, 5, 2.5, 1.25, \ldots$
4. $5, -5, 5, -5, 5, -5, \ldots$
5. $x, -2x^2, 4x^3$
6. $1, p, p^2, p^3, \ldots$
7. $4, -1, 0.25, -0.0625, \ldots$

Examples

1. Determine the 10^{th} and n^{th} terms of the following:

a) 3, 6, 12, 24, …

b) 40, -20, 10, -5, …

2. The second term of a geometric sequence is 4 and the 4^{th} term is 8. The common ratio is positive. Find the exact values of:

a) The common ratio.

b) The first term.

c) The 10^{th} term.

3. The numbers $3,\ x$ and $x+6$ form the first three terms of a positive geometric sequence. Find:

a) The value of x.

b) The 10^{th} term in the sequence.

Inequalities Example

What is the first term in the geometric progression $3,\ 6,\ 12,\ 24,\ \ldots$ to exceed 1 million?

Test Your Understanding

1. All the terms in a geometric sequence are positive.

The third term of the sequence is 20 and the fifth term 80. What is the 20^{th} term?

2. The second, third and fourth term of a geometric sequence are the following:

$$x, \quad x + 6, \; 5x - 6$$

a) Determine the possible values of x.

b) Given the common ratio is positive, find the common ratio.

c) Hence determine the possible values for the first term of the sequence.

Sum of terms of Geometric Series

Proof:

Examples

1. Find the sum of the first 10 terms of the following sequences

a)

$$3,\ 6,\ 12,\ 24,\ 48,\ \ldots$$

b)

$$4,\ 2{,}1,\ \frac{1}{2},\frac{1}{4},\frac{1}{8},\ \ldots$$

Example

Find the least value of n such that the sum of $1 + 2 + 4 + 8 + \cdots$ to n terms would exceed 2 000 000.

Test Your Understanding

The second and third terms of a geometric series are 192 and 144 respectively.

For this series, find

(*a*) the common ratio,

(2)

(*b*) the first term,

(2)

(*d*) the smallest value of n for which the sum of the first n terms of the series exceeds 1000.

(4)

Extension

MAT 2010 1B]

The sum of the first $2n$ terms of

$$1,\ 1,\ 2, \frac{1}{2},\ 4, \frac{1}{4}, 8, \frac{1}{8}, 16, \frac{1}{16}, \ldots$$

is

A) $2^n + 1 - 2^{1-n}$

B) $2^n + 2^{-n}$

C) $2^{2n} - 2^{3-2n}$

D) $\frac{2^n - 2^{-n}}{3}$

Divergence and Convergence

Sum to Infinity

Quickfire Examples: Calculate a, r and S_∞ for the following sequences

1. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

2. $27, -9, 3, -1, \ldots$

3. $p,\ p^2, p^3, p^4, \ldots \quad where\ -1 < p < 1$

4. $p,\ 1, \frac{1}{p}, \frac{1}{p^2}, \ldots$

Examples

1. The fourth term of a geometric series is 1.08 and the seventh term is 0.23328.

a) Show that this series is convergent.

b) Find the sum to infinity of this series.

2. For a geometric series with first term a and common ratio r, $S_4 = 15$ and $S_\infty = 16$.

a) Find the possible values of r.

b) Given that all the terms in the series are positive, find the value of a.

Test Your Understanding

6. The second and third terms of a geometric series are 192 and 144 respectively.

For this series, find

(*a*) the common ratio,
(2)

(*b*) the first term,
(2)

(*c*) the sum to infinity,
(2)

(*d*) the smallest value of n for which the sum of the first n terms of the series exceeds 1000.
(4)

Extension

1. [MAT 2006 1H] How many solutions does the equation

$$2 = \sin x + \sin^2 x + \sin^3 x + \sin^4 x + \cdots$$

have in the range $0 \leq x < 2\pi$

2. [MAT 2003 1F] Two players take turns to throw a fair six-sided die until one of them scores a six. What is the probability that the first player to throw the die is the first to score a six?

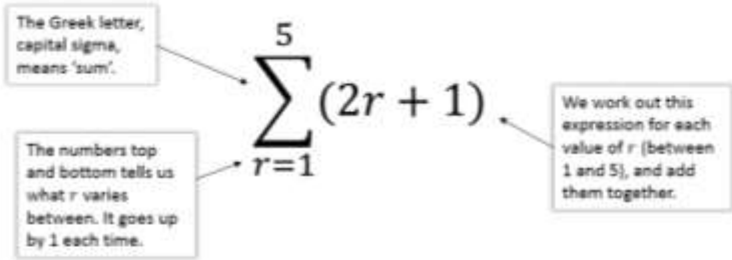
3. [Frost] Determine the value of x where:

$$x = \frac{1}{1} + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \cdots$$

(Hint: Use an approach similar to proof of geometric S_n formula)

Ex 3E Pg 75

Sigma Notation



	First few terms?	Values of $a,\ n,\ d\ or\ r$?	Final result?
$\sum_{n=1}^{7} 3n$			
$\sum_{k=5}^{15}(10-2k)$			
$\sum_{k=1}^{12} 5\times 3^{k-1}$			
$\sum_{k=5}^{12} 5\times 3^{k-1}$			

Test Your Understanding

Evaluate

$$\sum_{r=10}^{30} (7+2r).$$

"Use of Technology" Monkey says:
The Classwiz and Casio Silver calculator has a Σ button.

Try and use it to find:

$$\sum_{k=5}^{12} 2\times 3^{k}$$

Ex 3F Pg 77

Recurrence Relations

Example

6. A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1,$$
$$x_{n+1} = (x_n)^2 - kx_n, \qquad n \geq 1,$$

where k is a constant.

(*a*) Find an expression for x_2 in terms of k.
(1)

(*b*) Show that $x_3 = 1 - 3k + 2k^2$.
(2)

Given also that $x_3 = 1$,

(*c*) calculate the value of k.
(3)

(*d*) Hence find the value of $\sum_{n=1}^{100} x_n$.
(3)

Test Your Understanding

4. A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1,$$

$$x_{n+1} = a x_n + 5, \qquad n \geq 1,$$

where a is a constant.

(*a*) Write down an expression for x_2 in terms of a.

(1)

(*b*) Show that $x_3 = a^2 + 5a + 5$.

(2)

Given that $x_3 = 41$

(*c*) find the possible values of a.

(3)

Combined Sequences

Sequences (or series) can be generated from a combination of both an arithmetic and a geometric sequence.

Example

4. (i) Show that $\sum_{r=1}^{16}(3 + 5r + 2^r) = 131\,798$ **(4)**

(ii) A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_{n+1} = \frac{1}{u_n}, \quad u_1 = \frac{2}{3}$$

Find the exact value of $\sum_{r=1}^{100} u_r$ **(3)**

Extension

1. *[AEA 2011 Q3]* A sequence $\{u_n\}$ is given by

$$u_1 = ku_{2n} = u_{2n-1} \times p, \quad n \geq 1u_{2n+1} = u_{2n} \times q \qquad n \geq 1$$

(a) Write down the first 6 terms in the sequence.

(b) Show that $\sum_{r=1}^{2n} u_r = \frac{k(1+p)(1-(pq)^n)}{1-pq}$

$[x]$ means the integer part of x, for example $[2.73] = 2,\ [4] = 4$.

Find $\sum_{r=1}^{\infty} 6 \times \left(\frac{4}{3}\right)^{\left[\frac{r}{2}\right]} \times \left(\frac{3}{5}\right)^{\left[\frac{r-1}{2}\right]}$

2. [MAT 2014 1H] The function $F(n)$ is defined for all positive integers as follows: $F(1) = 0$ and for all $n \geq 2$,

$F(n) = F(n-1) + 2$ if 2 divides n but 3 does not divide ,

$F(n) = F(n-1) + 3$ if 3 divides n but 2 does not divide n,

$F(n) = F(n-1) + 4$ if 2 and 3 both divide n

$F(n) = F(n-1)$ if neither 2 nor 3 divides n.

Then the value of $F(6000)$ equals what?

3. [MAT 2016 1G] The sequence (x_n), where $n \geq 0$, is defined by $x_0 = 1$ and

$x_n = \sum_{k=0}^{n-1}(x_k)$ for $n \geq 1$

Determine the value of the sum $\sum_{k=0}^{\infty} \frac{1}{x_k}$

Classifying Sequences

A sequence is **strictly increasing** if the terms are always increasing, i.e. $u_{n+1} > u_n$ for all $n \in \mathbb{N}$.
e.g. $1, 2, 4, 8, 16, \ldots$

Similarly a sequence is **strictly decreasing** if $u_{n+1} < u_n$ for all $n \in \mathbb{N}$

A sequence is **periodic** if the terms repeat in a cycle. The **order** k of a sequence is **how often it repeats**, i.e. $u_{n+k} = u_n$ for all n.
e.g. 2, 3, 0, 2, 3, 0, 2, 3, 0, 2, … is periodic and has order 3.

Examples

For each sequence:

i) State whether the sequence is increasing, decreasing or periodic.

ii) If the sequence is periodic, write down its order.

a) $u_{n+1} = u_n + 3, \; u_1 = 7$

b) $u_{n+1} = (u_n)^2, \quad u_1 = \frac{1}{2}$

c) $u_{n+1} = \sin(90n°)$

Ex 3H Pg 82

Modelling

Examples

1. Bruce starts a new company. In year 1 his profits will be £20 000. He predicts his profits to increase by £5000 each year, so that his profits in year 2 are modelled to be £25 000, in year 3, £30 000 and so on. He predicts this will continue until he reaches annual profits of £100 000. He then models his annual profits to remain at £100 000.

a) Calculate the profits for Bruce's business in the first 20 years.

b) State one reason why this may not be a suitable model.

c) Bruce's financial advisor says the yearly profits are likely to increase by 5% per annum. Using this model, calculate the profits for Bruce's business in the first 20 years.

2. A piece of A4 paper is folded in half repeatedly. The thickness of the A4 paper is 0.5 mm.

(a) Work out the thickness of the paper after four folds.

(b) Work out the thickness of the paper after 20 folds.

(c) State one reason why this might be an unrealistic model.

Test Your Understanding

A company predicts a yearly profit of £120 000 in the year 2013. The company predicts that the yearly profit will rise each year by 5%. The predicted yearly profit forms a geometric sequence with common ratio 1.05.

(*a*) Show that the predicted profit in the year 2016 is £138 915.

(1)

(*b*) Find the first year in which the yearly predicted profit exceeds £200 000.

(5)

(*c*) Find the total predicted profit for the years 2013 to 2023 inclusive, giving your answer to the nearest pound.

(3)

Extension

AEA 2007 Q5

The figure shows part of a sequence $S_1,\ S_2,\ S_3, \ldots$, of model snowflakes. The first term S_1 consist of a single square of side a. To obtain S_2, the middle third of each edge is replaced with a new square, of side $\frac{a}{3}$, as shown. Subsequent terms are added by replacing the middle third of each external edge of a new square formed in the previous snowflake, by a square $\frac{1}{3}$ of the size, as illustrated by S_3.

a) Deduce that to form S_4, 36 new squares of side $\frac{a}{27}$ must be added to S_3.

b) Show that the perimeters of S_2 and S_3 are $\frac{20a}{3}$ and $\frac{28a}{3}$ respectively.

c) Find the perimeter of S_n.